The analysis and projection of mortality rates for annuity and pensions business

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I hereby declare that the work presented in this thesis was carried out by me while a student at Heriot-Watt University, Edinburgh, except where due acknowledgement is made, and has not been submitted for any other degree.

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Date

If the proper study of mankind is man,
we can claim that the proper study of
actuaries is mortality.

A. R. N. Ratcliff, President of the
Faculty of Actuaries (in Benjamin, 1982)

Abstract

Longevity risk is a major issue for the developed world. As both mortality rates and birth rates fall, the increasing burden of providing for retirees falls on a smaller working population. Under such circumstances, the accurate modelling and measurement of longevity risk becomes particularly important.

Longevity risk is present in the annuity portfolios of insurance companies, and increasingly of reinsurers as well. However, the biggest concentration of longevity risk in the private sector in the United Kingdom is most often in the shape of defined-benefit pension promises by employers. This makes longevity risk of crucial interest to managers and investors, even if they think that their business has nothing to do with insurance.

Actuaries handle longevity risk by breaking it into two components: the current (or period) rates of mortality, and the projection of future rates. In both areas actuaries have made significant advances in their modelling and understanding of longevity risk. This critical review outlines how methods have developed, and how the papers in the accompanying thesis have contributed to these advances.

Dedication

This critical review and the accompanying thesis are dedicated to the memory of my mother, Elizabeth A. Richards, and to her grandchildren, Robert and Eleanor Richards, who belong to a generation which will be paying for the pensions of previous generations for much longer than anyone ever imagined.

Acknowledgements

The author is indebted to Dr Iain Currie, co-author of two of the papers in the thesis, and also to Professor Angus Macdonald, for suggesting that this thesis could be attempted.

The author would also like to thank Gavin Ritchie, his business partner since 2006 and implementor of the database code for the deduplication algorithms described in Richards (2008b).

This document has been typeset in LaTeX and all graphs were generated using R (2011).

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Chapter 1

Introduction

This critical review accompanies the PhD thesis submitted under Regulation 43 "Degree of Doctor of Philosophy by Published Research", Heriot-Watt University. The purpose of this review is to show the following (with references to Regulation 43 in parentheses):

- Outline the purpose behind the submitted works (¶9.4.1).
- How the published papers form a coherent body of work (¶9.4.2).
- How the papers relate to each other (¶9.4.2).
- The candidate's contribution to those papers which are jointly authored (¶9.5).
- The extent to which the papers have undergone critical peer review.
- What impact the body of work has had in terms of (i) academic impact, (ii) professional impact, and (iii) commercial impact.

Each of the six papers contained in the thesis examines one or more aspects of the measurement and management of longevity risk. A summary of how the papers relate to each other and form a coherent body of work is given in Chapter 4. The candidate's contribution to jointly authored papers is described in Chapter 15.

The results have had practical applications in pension schemes and the management of annuity portfolios, as described in Chapters 18 and 19.

Chapter 2

Background

> By providing financial protection against the major 18th- and 19th-century risk of dying too soon, life insurance became the biggest financial industry of that century [...] Providing financial protection against the new risk of not dying soon enough may well become the next century's major and most profitable financial industry.
>
> Drucker (1999)

There is a long tradition of private pension provision and private annuities in the United Kingdom — see Pensions Commission (2004) for a detailed history. As a consequence there is a very large exposure to longevity risk in the private sector — see Richards and Jones (2004) and also Lane, Clark and Peacock (2010). Most of this risk is managed under the advice of actuaries, who therefore form a natural target audience for research on understanding and managing longevity risk.

A pension scheme promises an income in retirement until death. A life-office annuity guarantees the scheduled income until death. In each case the liability is defined by just two items: the amount to pay and how long the recipient will live. The amount to pay is usually fixed as a nominal cash amount, but discretionary pension increases were often awarded in pension schemes during periods of high inflation, and it is possible for pensions or annuities to be linked to some external index. How long the recipient will live is therefore a key variable, but this is fundamentally unknown. This is longevity risk.

There are many other aspects to managing a pension fund or annuity portfolio, including how to invest the backing assets (if there are any) and any credit risk, reinvestment risk or asset-liability mismatch there might be. However, such risks concern the *funding* of the pension or annuity, not the nature of the promise or guarantee itself. Although these other risks may often take on greater financial significance at times, it is longevity risk which is the core risk in defining the liability itself (leaving aside the question of indexation). We thus distinguish between the liability (the promise to pay a pension until death) and the value placed on that liability for funding and management purposes. This thesis is concerned with longevity risk in isolation; in order to avoid distraction from questions of funding liabilities, illustrations of value will make the simplifying assumption of a constant interest rate for discounting.

Chapter 3

Why worry about longevity risk?

Actuaries' interest in longevity risk is not new, as this comment from the discussion of a paper on mortality in 1956 shows:

> The actuary's interest in the trend of mortality has taken on a more pressing character in recent years, for the trend at the older ages has become one of the great actuarial problems of the immediate future.
>
> A. Pedoe in Gwilt (1956), page 167

However, actuaries' interest in longevity has waxed and waned over the intervening decades. Simply put, actuaries' interest in mortality and longevity is inversely correlated with the yields obtained on gilts and corporate bonds. This is illustrated in Figure 3. The left panel shows how gilt yields have fallen since 1984, while the centre panel shows how a specimen annuity factor to a male aged 65 increases as the gilt yield falls. The yields in the left-hand panel have fallen to a third of their level in the mid-1980s, and the liability values in the centre panel have roughly doubled as a result. As the cost of providing a level annual pension has doubled, so pension schemes have closed their doors: first to new entrants, and latterly to future accrual of benefit for existing members (Pensions Commission, 2004). These are questions of cost and funding, however, and not of longevity risk *per se*.

The left and centre panels of Figure 3 simply show that annuity factors go up as yields go down. A less appreciated aspect of low interest rates is the increased

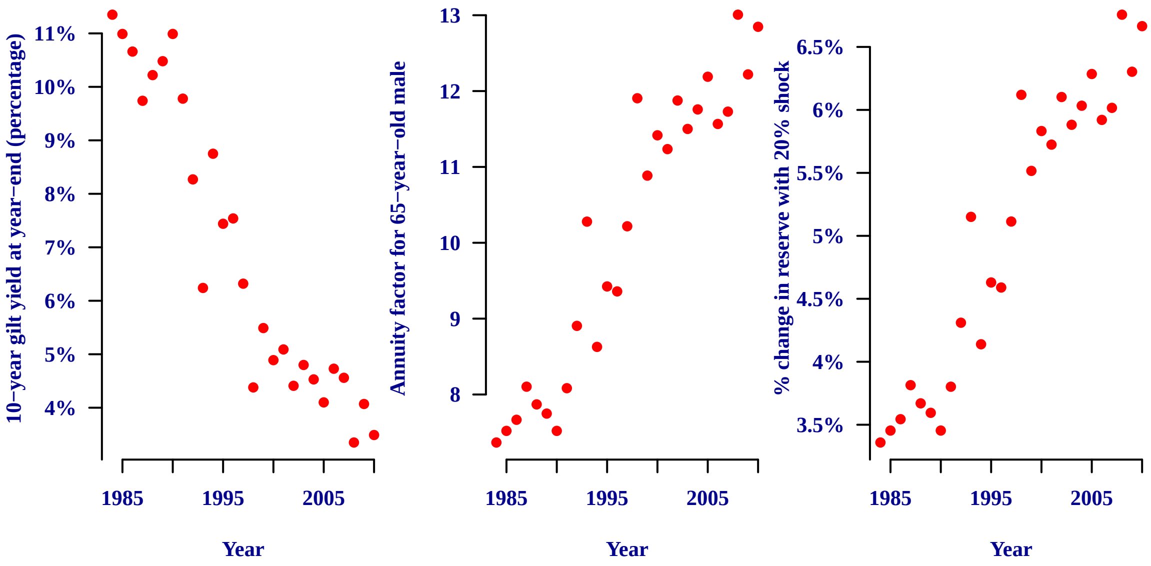

Figure 3.1: Gilt yields (left), annuity factors for males aged 65 (centre) and sensitivity of those anuity factors to a 20% fall in absolute mortality rates (right). Source: End-year yields from British Government Stock (10-year nominal par yield, series IUAMNPY from Bank of England) and own calculations for single-life continuous annuity factor for male aged 65 using the same yields and mortality according to the table S1PA (CMI, 2008).

sensitivity of annuity factors (and thus annuity and pension reserves) to unexpected changes in mortality. This increased sensitivity is a key item of interest for regulators, investors and pensioners. The right-hand panel of Figure 3 shows the percentage change in reserves due to an immediate 20% reduction in mortality levels[1]. Figure 3 shows one reason why actuaries and regulators are a lot more focused on longevity risk nowadays: as interest rates have fallen, the sensitivity of annuity reserves to a longevity shock has doubled (all other things being equal). At the time of writing, interest rates are at an historic low in the UK — see Figure 3 — which means the sensitivity of pension-scheme liability values to changes in longevity assumptions is particularly pronounced.

However, longevity risk is not merely a technical concern for actuaries or pension funds. On the contrary, it poses some fundamental challenges for society in terms of industry, employment and public policy (Pensions Commission, 2004). To see why, we must first consider the differing treatment of longevity risk in insurance companies

[1]The choice of a 20% drop in mortality rates comes from the QIS5 rule for reserving for annuities under Solvency II. It is the proposed longevity shock which annuity reserves must be able to withstand under the so-called “standard formula” approach — see European Commission (2010).

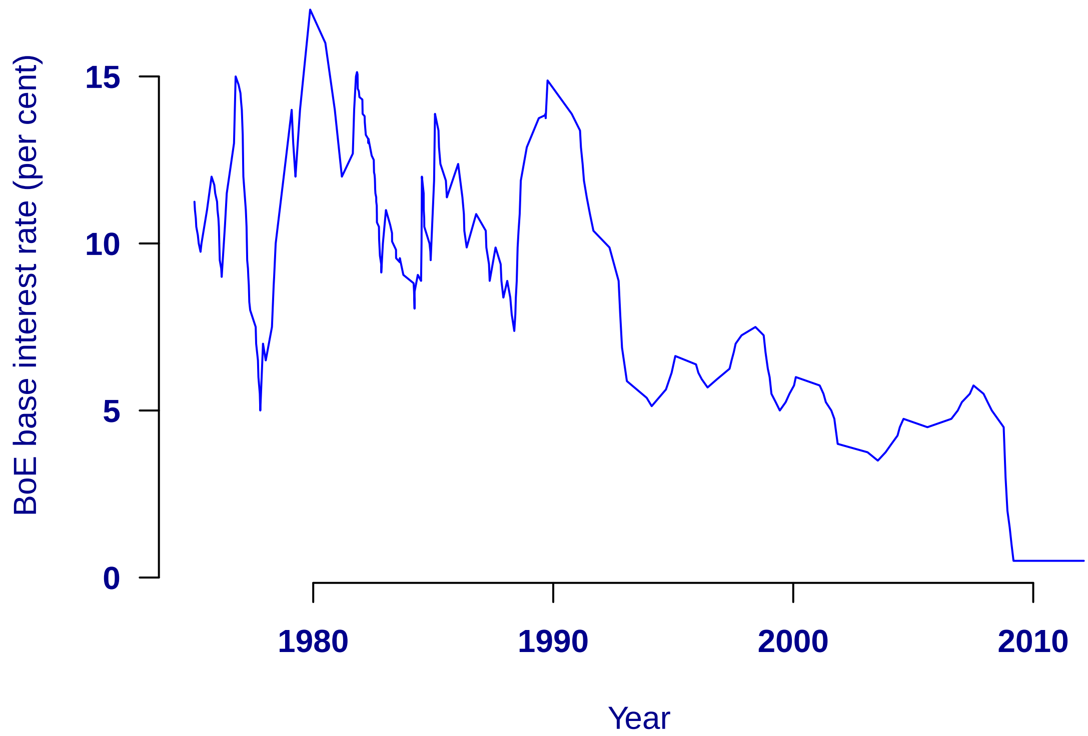

Figure 3.2: Bank of England base rates since 1975. Source: Bank of England, series IUDBEDR, accessed 10th February 2011.

and pension funds.

In the UK, and indeed throughout the European Union and elsewhere, longevity risk in company pension schemes is regulated in a quite different manner from longevity risk in insurance companies. In particular, pension schemes at the time of writing are often underfunded — Lane, Clark and Peacock (2010) showed that just five companies in the FSTE 100 Index had asset values equal to or greater than the value of the liabilities in their pension schemes. Of the rest, many have had a pension-scheme deficit for several years and seem likely to run one for some years to come. In contrast, insurers are subject to a requirement not just to hold prudent mathematical reserves, but also a solvency margin on top of this. Any insurer unable to demonstrate solvency would be promptly closed to new business. This separate treatment of pension schemes and insurers may look a little odd to the casual observer, as some companies have pension schemes as big as the longevity liabilities of some of the biggest insurers. For example, Table 3.1 shows four selected companies listed in the FTSE-100 Index, each of which is exposed to a large amount of longevity risk in the form of either a pension scheme or annuity portfolio (or both). Despite this, only one of the companies (Prudential plc) is legally a life insurer and thus subject to the much stricter regulations of life-company reserving.

Table 3.1: Longevity liabilities for four selected UK-listed companies. Source: 2009 pension-scheme liabilities from Lane, Clark and Peacock (2010), plus Form 14 liabilities for end-2009 from Prudential (2010a, 2010b).

Company	Longevity liabilities
Royal Dutch Shell	£38.8 bn
Prudential plc	£33.4 bn
BT	£33.3 bn
Royal Bank of Scotland	£30.8 bn

One reason given for this separate treatment is that a pension scheme is not the core business of the employer, and that the employer should therefore be allowed to focus on managing the business and not devote excessive management attention to a non-core activity like future staff benefits. Table 3.2 shows that this this is not always true. In the case of British Airways, the *Financial Times* described the company not as an airline but as “a leveraged investment trust with a troublesome sideline in air travel”.

Table 3.2: Longevity liabilities related to market capitalisation of four selected UK-listed companies. Source: 2009 pension-scheme liabilities and market capitalisations from Lane, Clark and Peacock (2010).

Company	Longevity liabilities	Market capitalisation	Liabilities / Market cap.
British Airways	£12.8 bn	£1.6 bn	791%
BT	£33.2 bn	£6.0 bn	551%
Invensys	£4.8 bn	£1.3 bn	364%
Royal Bank of Scotland	£30.8 bn	£16.6 bn	186%

In some cases, pension schemes have even become an existential threat to the parent company, and thus a threat to the employment it provides. This is illustrated in Table 3.3, which shows how the market values of some listed companies can be overshadowed by the scale of the funds requiring to be injected into the pension scheme.

Tables 3.2 and 3.3 suggest that some companies are being increasingly run to support their pension funds. This phenomenon was predicted by Drucker (1976):

Table 3.3: Unfunded longevity liabilities related to market capitalisation of three selected UK-listed companies. Source: 2009 pension-scheme deficits and market capitalisations from Lane, Clark and Peacock (2010).

Company	Deficit	Market capitalisation	Deficit / Market cap.
BT	£4.0 bn	£6.0 bn	66%
BAE Systems	£5.6 bn	£12.7 bn	44%
British Airways	£0.6 bn	£1.6 bn	37%

> The "means of production" [...] is being run for the benefit of the country's employees. Profits increasingly become retirement pensions, that is, "deferred compensation" of the employees.
>
> Drucker, P. (1976)

A dramatic example of this came in February 2011 when the food-processing company Uniq "acknowledged that the company's size was insufficient to satisfy its pension deficit, and agreed to cede 90 per cent of its shares to the pension scheme" — Financial Times (2011) and Uniq (2011). Since defined-benefit pension schemes are increasingly closed to new members, or even future accrual, Drucker's maxim might need updating — companies will increasingly be run for the benefit of *past* employees and not current ones. The thesis accompanying this review is about the mathematical modelling of mortality and longevity. However, it is worth bearing in mind the considerable socio-political role that pension schemes — and the longevity risk therein — play in modern society.

There is an unsettling circuitousness to the funding of some pension schemes. Both an insurance company and a pension scheme must hold sufficient assets to be able to pay pensions when they fall due. In the case of pension schemes, many hold some of their investments in the form of equities. However, it would be bad risk management to hold assets whose value might be negatively correlated with longevity risk, since this would expose the scheme to a double source of deficit: increased liabilities due to unexpected increases in longevity, but also reduced equity values for those companies with substantial longevity exposure themselves. Partly as a reaction to this, pension funds increasingly invest in government bonds, which binds both pension funds and governments in a state of mutual dependency:

> What happens if governments default on their bonds, or inflate away the debts, rather than put their voters through the years of austerity required to pay them down? The result would be huge shortfalls in domestic pension funds.
>
> — The Economist (2011)

One could equally turn this question around: after the credit crisis of 2008, who better to buy the increasing volume of government bonds than pension funds seeking matches to long-term liabilities?

In January 2012 Shell closed its pension scheme to new members, meaning that none of the FTSE-100 companies offered a final-salary pension to new staff. Defined-benefit pension schemes elsewhere in the UK are increasingly closed to new members, or even future accrual (Pensions Commission, 2004). As a result, pension schemes in the United Kingdom are set to "age" at a faster rate than ever before. In a direct parallel with the ageing of society — a higher support ratio of retired people to current workers — pension schemes face a rapidly increasing ratio of pensioners to active and deferred members. In both cases — an ageing society or an ageing pension fund — the understanding, measurement and management of longevity risk lies at the core of the response.

Chapter 4

Summary of papers in thesis

This chapter briefly summarises the content of the six papers in the main thesis, and illustrates the contribution to new knowledge in each case.

Richards, Kirkby and Currie (2006) present the then-new idea of smoothing a two-dimensional P-spline model of mortality by age and year of birth (cohort), instead of the more-usual smoothing by age and period. These two contrasting approaches are used to show that year-of-birth patterns ("cohort effects") in England & Wales data are evident with either method, but that the age-cohort model fits the data better. The paper shows how P-spline models represent a more sophisticated approach to smoothing by moving average or by kernel smoothers.

Richards et al (2007) take the work of Richards, Kirkby and Currie (2006) and apply both age-period and age-cohort models to the mortality experience of seven industrialised nations. The seven nations were chosen for their widely differing social histories in the latter half of the twentieth century, and the authors found that not all countries exhibited as strong a cohort effect as England & Wales did. The authors illustrated some of the practical difficulties which can be encountered when applying two-dimensional P-spline models to such data, and also examined changes in the patterns of causes of death in each of the seven countries. Consideration was given to the challenges in explaining projected all-cause mortality improvements in terms of reductions in mortality due to specific causes of death.

Richards (2008a) documents the differing quality of data in population mortality statistics in England & Wales — deaths are recorded continuously, and death counts are highly reliable, whereas population figures are merely estimates between relatively widely spaced censuses. Specific concerns are raised about the quality of UK population estimates, and methods are presented for detecting mortality changes using the death data only.

Richards (2008b) presents a framework for applying survival models to pensioner mortality data. Left-truncated observations are a much bigger issue for actuaries than for some other users of survival models, and solutions are given for six different parametric models of mortality. Richards (2008b) documents the additional data-preparation steps required by actuaries, as duplicate records are much more common in actuarial work than in other research disciplines. The main innovations are the scheme for deduplication, the derivation of formulae for left-truncated integrated hazard functions, and the proof of how the Makeham-Beard mortality law can be derived from a simple reliability model in engineering terms.

Richards and Currie (2009) present a smoothed variant of the Lee-Carter model. The main innovation is the use of a penalty forecast for the time-varying element, κ, in place of the more common time-series approach. The critical importance of model risk is demonstrated by applying two slightly different models to the same data set, which produce different projections with different statements of uncertainty. The impact of this model risk is illustrated by applying the projections to three different portfolios of pensions in payment. The paper not only demonstrates model risk, but also the varying impact of idiosyncratic risk and concentration risk on each portfolio.

Richards (2010b) extends Richards (2008c) and presents solutions to the left-truncation problem for sixteen different parametric survival models. The effectiveness of the sixteen models is compared for the mortality experience of a large data set of life-office annuitants. A map of the relationships between the sixteen models is presented, and a rationale is given for a common approach to parameterising the models.

Chapter 5

Notation

The notation used througout the papers, and especially in Richards (2008b, 2010b), is briefly described here. The probability of a life aged x dying before age $x+1$ is denoted q_x, i.e.

$$q_x = \Pr\left(\text{death before age } x+1 | \text{alive at age } x\right), \qquad x \geq 0 \tag{5.1}$$

q_x is spoken of as a mortality rate, but here it is also the parameter of a Binomial model for the number of deaths, D_x, which occur amongst n_x identical lives aged x, i.e.

$$D_x \sim B(n_x, q_x) \tag{5.2}$$

The probability concept can be extended to an arbitrarily small interval of time, h, such that:

$${}_hq_x = \Pr\left(\text{death before age } x+h | \text{alive at age } x\right), \qquad h > 0 \tag{5.3}$$

By letting h tend towards zero from above we define the concept of the *instantaneous hazard rate*, μ_x, which is defined as:

$$\mu_x = \lim_{h \to 0^+} \frac{{}_hq_x}{h} \tag{5.4}$$

The quantity μ_x in Equation 5.4 is referred to by actuaries as the *force of mortality* (Neill, 1986), and it is also the same concept described by engineers as the *failure rate*

(Gavrilov and Gavrilova, 2001).

The probability of a life aged x surviving at least t years is denoted ${}_tp_x$, and it is related to q_x and μ_x as follows:

$$ {}_tp_x = \prod_{s=0}^{t-1} \left(1 - q_{x+s}\right), \qquad \text{for integral } t \tag{5.5} $$

$$ {}_tp_x = \exp\left(- \int_0^t \mu_{x+s} ds\right), \qquad t \geq 0 \tag{5.6} $$

${}_tp_x$ is the *survivor function* (Collett, 2003) and it is tightly connected to the idea of the future lifetime of an individual aged x being a random variable, T_x. T_x has a distribution function, $F_x(t)$, defined as follows:

$$ F_x(t) = 1 - {}_tp_x \tag{5.7} $$

In the case where T_x is a continuous random variable, the probability density function, $f_x(t)$, is defined as:

$$ f_x(t) = {}_tp_x \mu_{x+t}, \qquad t \geq 0 \tag{5.8} $$

The quantity $f_x(t)$ in Equation 5.8 is referred to by actuaries as the *curve of deaths* (Beard, 1959).

Chapter 6

Splines

Four of the six papers in the thesis make use of splines for fitting smooth-but-flexible curves to mortality data. This chapter explains in detail what splines are, how they are constructed and how they form a flexible regression basis for mortality modelling.

Collins English Dictionary defines a spline as follows:

> [...] **2.** a long narrow strip of wood, metal etc.; slat. **3.** a thin narrow strip made of wood, metal or plastic fitted into a groove in the edge of a board, tile, etc., to connect it to another.
>
> Collins (1986)

Splines in the physical sense have been used by draughtsmen seeking to draw smooth curves through a finite number of fixed points. Splines in the abstract sense have been widely used elsewhere: modern computer graphics use splines extensively for their efficient and economical representation of organic forms. These can be animations, or for the modern scaleable fonts used in typesetting systems such as the one used to create this document — see Knuth (1986). de Boor (2001) published a number of results regarding splines, including a recurrence relation which greatly simplified their evaluation. de Boor's results are widely used in many fields, including the design of modern airplanes:

> Boeing uses de Boor's recurrence relation to perform something like 500 million spline evaluations every day
>
> ---
>
> Grandine (2005)

The use of splines does not stop at computer-aided design. Another important application of splines at Boeing lies in calculating optimal orbit and flight trajectories. The continuous variables representing the physics of the vehicle and its controls are replaced by spline approximations, an illustration of which is given in Grandine (2005).

Splines have long been used in actuarial work: actuarial tables since the 1970s have been graduated using splines, and their use in smoothing was discussed in Barnett (1985). The key feature of splines — flexible yet economical representation of curves — is immediately applicable to the very essence of actuarial work: the mortality curve. When this flexibility is combined within a statistical framework, an actuary can balance the flexibility of fit with the degree of curviness justified by the mortality data.

A spline in the mathematical sense is a series of polynomials joined together at *knot points*. The order of the polynomials is connected to the number of knots: a spline of order m will have $m+1$ polynomials of order m joining $m+2$ knot points. Outside of these knot points the spline takes the value zero. This is illustrated in Figure 6. A spline is therefore a local function, since it evaluates to zero for all but a small part of the real line.

A basis of overlapping splines is created such that each spline passes through the join points of neighbouring splines. This is illustrated in Figure 6, which shows how spline heights in a regression are not independent of each other due to the overlap each has with its neighbours. It is this overlap which gives splines many features in common with a moving average (Richards, Kirkby and Currie, 2006). One point to note is that splines overlap more as their degree increases: a spline of degree 1 has just two neighbouring splines which pass through it, whereas a spline of degree 3 has six neighbouring splines which pass through it. In general a spline of degree m has $2m$ overlapping splines. In order to get a satisfactory degree of smoothness, it is necessary to pick a spline degree resulting in a reasonable amount of overlap. In the papers forming the thesis, cubic splines have been used, i.e. splines of degree 3.

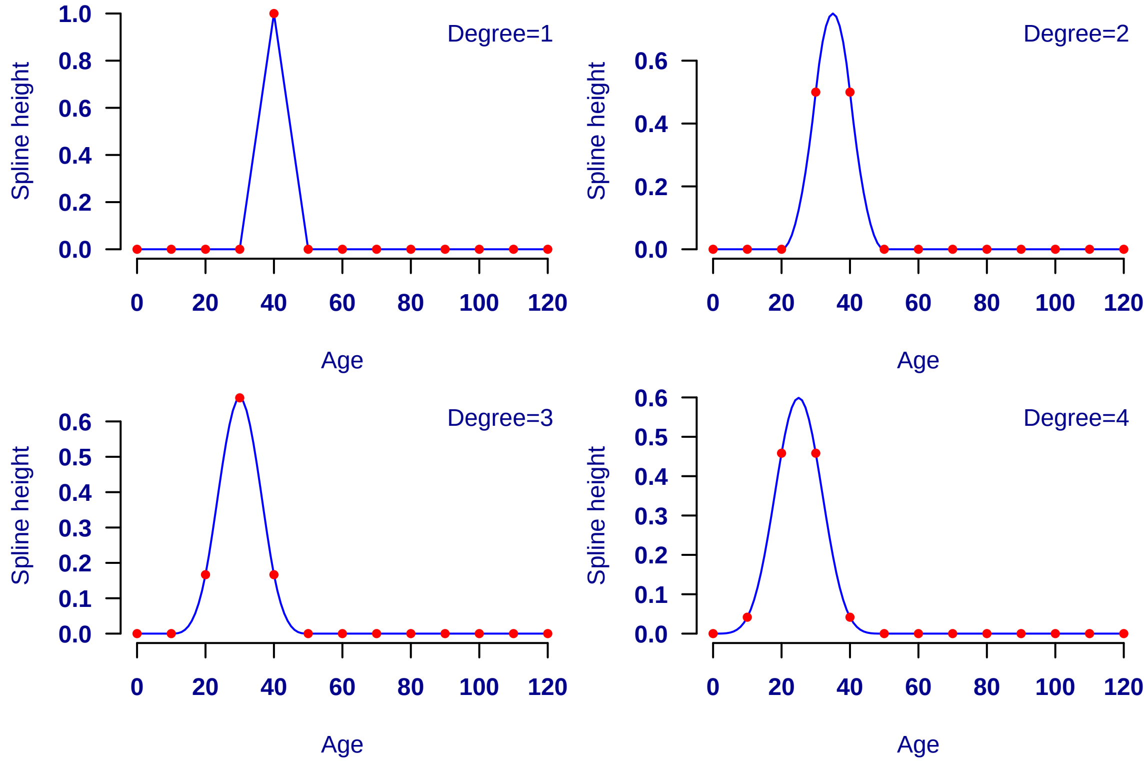

Figure 6.1: Splines of varying degrees.

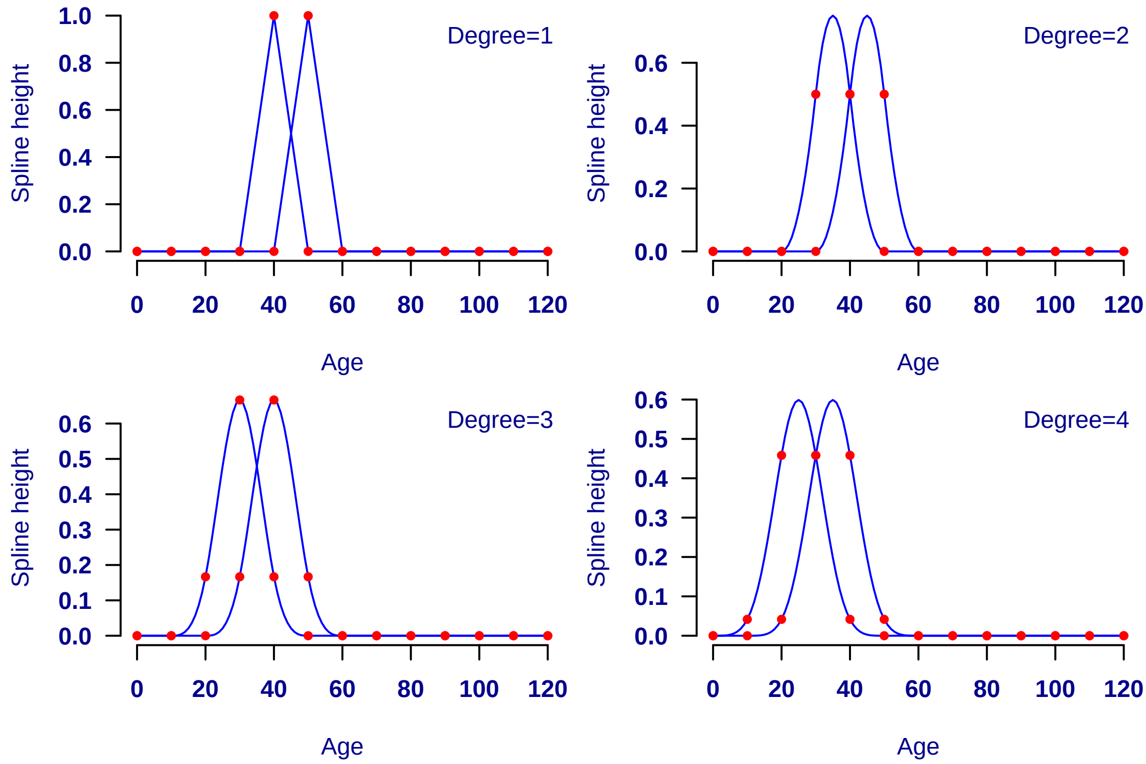

Figure 6.2: Splines of varying degrees demonstrating overlapping nature by passing though join points of neighbouring spline. Only two splines in each basis are shown for clarity.

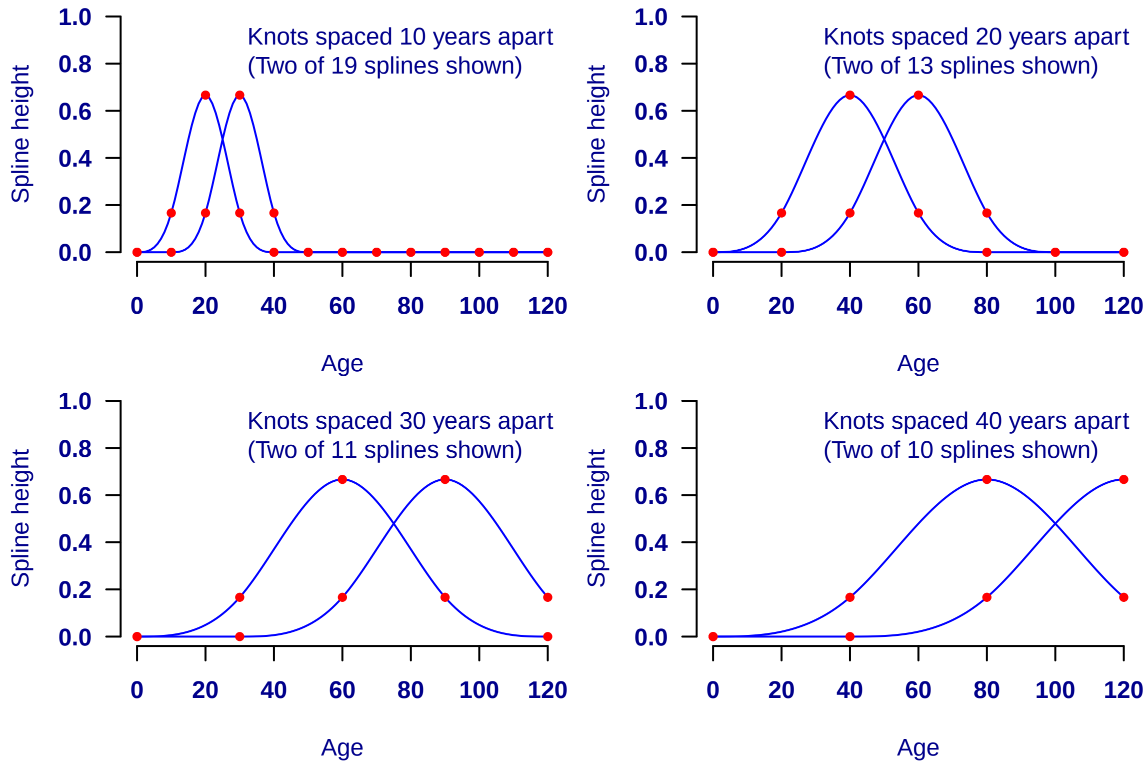

Figure 6.3: Cubic splines showing the impact of different choices of knot spacing.

As the knot spacing increases, the number of splines required in the basis to span the regression domain decreases. This is shown in Figure 6. One consequence of this is that a basis with widely spaced knots requires less smoothing than a basis with a narrower spacing. Flexibility also reduces with increasing knot spacing, although this can be an advantage when one wants to place less reliance on a smoothing parameter.

Figure 6 shows a full basis of cubic splines spanning the age domain [0, 120] required for regression problems involving human mortality rates. A knot spacing of ten years has been used, and the knot points range from -30 to +150 in ten-year increments. This wider range is caused by the requirement for partial splines centred outside the regression domain [0, 120], but which are needed to overlap with splines wholly inside the domain.

We now illustrate how splines are used in a mortality regression problem. We assume we have a vector of death counts, d_x, where x is the age last birthday. We further assume that we have corresponding central exposed-to-risk data, $e_{x+\frac{1}{2}}$, i.e. mid-age population estimates. We assume that the number of deaths is a random variable, D_x, with a Poisson distribution, i.e.

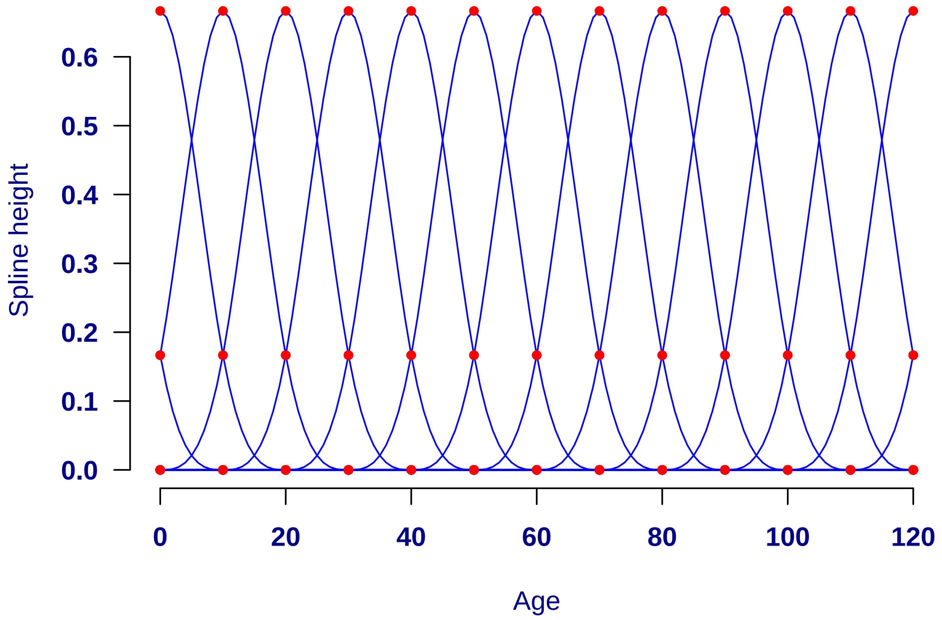

Figure 6.4: Full basis of cubic splines, including splines centred outside the regression domain of [0, 120].

$$D_x \sim \text{Poisson}\left(e_{x+\frac{1}{2}} \times \mu_{x+\frac{1}{2}}\right) \tag{6.1}$$

where $\mu_{x+\frac{1}{2}}$ is the force of mortality (hazard rate) applying at age $x + \frac{1}{2}$. For flexibility we use a basis of m B-splines, as shown in Figure 6. The force of mortality is therefore specified as follows:

$$\log \mu_{x+\frac{1}{2}} = \sum_{j=1}^{m} \theta_j B_j \left(x + \frac{1}{2}\right) \tag{6.2}$$

where $B_j\left(x + \frac{1}{2}\right)$ is the j^{th} B-spline evaluated at $x + \frac{1}{2}$ and the θ_j are coefficients to be estimated. Richards, Kirkby and Currie (2006) give a worked example of how the heights of the B-splines are varied by the θ_j and how the products of the B_j and θ_j are summed to form the value of $\log \mu$. By working on a logarithmic scale, the θ_j are free to vary across the real line. Equation 6.2 is directly analogous to the likes of the three-component mortality law proposed by Heligman and Pollard (1980). The difference is that Heligman and Pollard's component functions apply across the entire age range, whereas the splines in Equation 6.2 are purely local functions which have zero value two knot points distant from their centre. Heligman and Pollard's mortality law is designed with a specific demographic interpretation for each component: (i) decreasing childhood mortality, (ii) the so-called "accident hump" of young adults, and (iii) increasing mortality in later adult life. No such interpretation is intended for each spline.

To estimate the θ_j we form the likelihood function for maximisation:

$$L \propto \prod_x \frac{1}{d_x!} \left(e_{x+\frac{1}{2}} \times \mu_{x+\frac{1}{2}}\right)^{d_x} \exp\left(-e_{x+\frac{1}{2}} \times \mu_{x+\frac{1}{2}}\right) \tag{6.3}$$

although in practice we would maximise the log-likelihood function, l, defined as follows after dropping additive constants involving data only:

$$\ell = \sum_x d_x \log \mu_{x+\frac{1}{2}} - \sum_x e_{x+\frac{1}{2}} \times \mu_{x+\frac{1}{2}} \tag{6.4}$$

If the spacing between the number of splines is too small we could have an erratic pattern of the θ_j, so we adapt Equation 6.4 as follows:

$$\ell^p = \ell - \lambda P(\theta) \tag{6.5}$$

where $P(\theta)$ is a *penalty function* to penalise roughness in the θ_j and λ is a parameter controlling the degree of smoothing applied (see Eilers and Marx, 1996). A common example is to use a second-order penalty function, such the following:

$$P(\theta) = (\theta_1 - 2\theta_2 + \theta_3)^2 + \ldots + (\theta_{m-2} - 2\theta_{m-1} + \theta_m)^2 \tag{6.6}$$

The expression ℓ^p is known as a *penalized log-likelihood* and, for a given value of λ, the maximum penalized log-likelihood estimates of the θ_j are given by maximising Equation 6.5. The value of λ can either be pre-set, or else selected by picking the value of λ which minimises an information criterion, such as the AIC, BIC or GCV (Akaike, 1987). However, optimizing an information criterion for the fit of a model to the data will not necessarily produce a good projection model, as illustrated in Chapter 10. For this reason an alternative approach is to optimize the value of λ for a given number of degrees of freedom in the model.

Besides giving a smooth fit to the data, a further advantage of using penalised splines is that they can be used to extrapolate outside the data range. Figure 6 shows how the penalty function has been used to adjust the height of splines with centres outside the data range such that sensible extrapolated values can be created for high ages where data is unavailable. This is a particularly useful feature for actuaries, who need mortality rates at the very oldest ages in order to complete their calculations for pension-fund valuations and annuity portfolios.

The source code for producing the regression and chart in Figure 6 is freely available at www.longevitas.co.uk/graduate

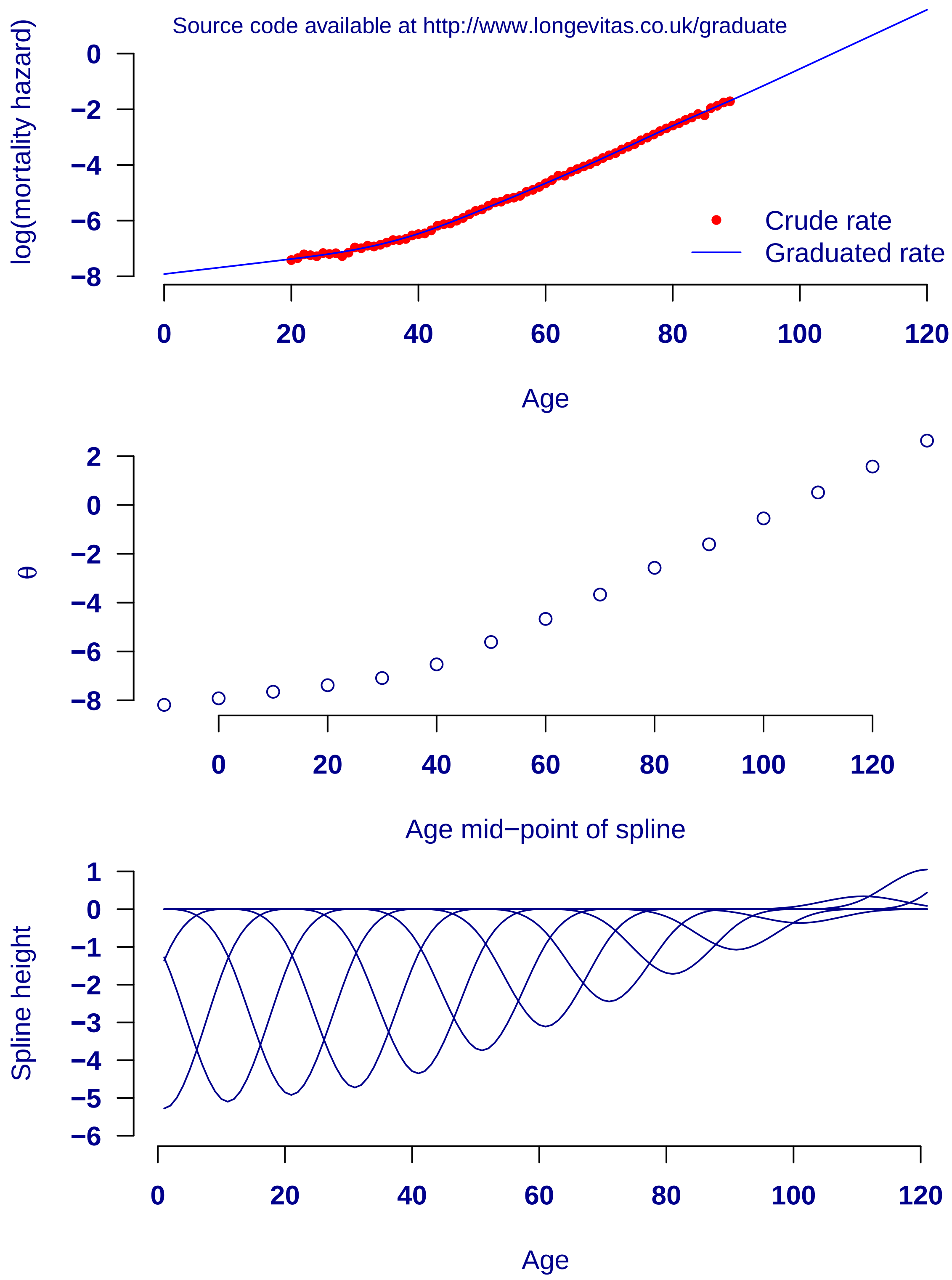

Figure 6.5: Top panel: crude force of mortality for males in England & Wales in 2004, together with fitted and extrapolated values using the spline basis in Figure 6; middle panel: regression coefficients, θ, from Equation 6.2; lower panel: splines from Figure 6 as adjusted by the regression coefficients.

Chapter 7

Components of longevity risk

Actuaries divide their treatment of longevity risk into two components: (i) current mortality rates, especially differentials between sub-groups, and (ii) future mortality rates. With a suitably sized portfolio with credible mortality experience, the determination of current (or recent) mortality rates is a question of measurement. In contrast, future mortality rates are a question of projection. In both cases statistical models are used, but these models tend to have very different structures.

The modelling of a portfolio's recent mortality experience is used not only to determine current mortality rates, but also to detect and measure mortality differentials. The rationale for this varies, but most commonly it is a question of underwriting, i.e. knowing what level of mortality to assume for the purpose of pricing or risk transfer. The goal of the model is to accurately and parsimoniously represent the mortality differentials exhibited in the portfolio. Typically, rich data is available for each individual life, and risk modelling can take place at the level of the individual in a survival model (Richards, 2008b and 2010b).

In contrast, very few portfolios have enough historical data to calibrate a model for projection. Actuaries and others are invariably forced to use other, unrelated data sets which do have this historical data. However, such data is typically much less rich, often with data split only by age and gender. This kind of data is also invariably only available for grouped counts, not for individuals. When building a model for mortality, or any other kind of risk, there are a number of known issues in the modelling process which actuaries have to consider:

- Model risk. You do not actually know what model structure is most appropriate for your portfolio or risk. This is particularly keenly felt for mortality projections.

- Basis risk. Even if you knew the correct model, you would have to calibrate it using the same population you want to model. However, if you fit a model to the experience from one portfolio (or population), yet use it to assess the risk in a second, you run the risk that the model is not transferable. A common example is the building of projection models using population data and then applying them to a specific portfolio.

- Parameter risk. Even with a good data set for your portfolio and a good model to fit, your parameter estimates are still subject to uncertainty as data are finite.

- Idiosyncratic risk. Even if your model were correct and had minimal parameter risk, you cannot predict precisely when a given individual will die.

- Concentration risk. Linked to idiosyncratic risk, the cost of uncertainty over when an individual dies is magnified in financial significance if that person has an unusually large benefit.

Each of these risks in the modelling process is potentially measurable or manageable:

- In the case of model risk, the obvious solution is to try a variety of models. This cannot eliminate model risk, but it does reduce the adverse consequences of relying on a single model which will amost inevitably turn out to be wrong — see Richards and Currie (2009) for an illustration with three pensioner populations.

- Basis risk is best dealt with by building and calibrating a model to the experience data of the portfolio itself. If this is not possible, say because a projection model is based on population statistics, then an explicit reserve will need to be held for basis risk. Alternatively, a so-called *piggyback model* can postulate a simple link between the population of interest and the reference population.

- Parameter risk can be explored by varying a parameter in a way consistent with the estimated standard error. The rest of the model can be refitted subject to this parameter being fixed, and the impact tested on the valuation of liabilities. An illustration of this was done in Richards (2009).

- Idiosyncratic risk is best explored by simulating the entire portfolio in run-off. This must be done on a life-by-life basis, as selecting a handful of model points to "represent" the portfolio will not help. See Richards and Currie (2009) for details and results for three different portfolios.

- Concentration risk is also best explored by simulation, again on a life-by-life basis. A handful of model points cannot summarise the rich diversity of benefits and risk profiles in a portfolio.

Richards and Currie (2009) carried out run-off simulations of three portfolios of differing sizes, demonstrating that the relative roles of parameter (trend) risk, idiosyncratic risk and concentration risk were portfolio-specific.

Chapter 8

Methods for analysing period mortality experience

The traditional actuarial approach to investigating mortality is to compare experience data against a pre-existing standard table of mortality rates by age and (usually) gender. A ratio of actual deaths against expected deaths would be calculated as follows:

$$\frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} q_i} \tag{8.1}$$

where n is the number of individuals, q_i is the probability of death for individual i according to a standard table, and d_i is an indicator variable taking the value 1 is life i is dead and zero otherwise. However, different lives have very different policy sizes, and it is known that life expectancy varies by socio-economic group, as shown in Figure 8. For actuarial work it is therefore critical to allow for socio-economic differentials, as the tendency for wealthier individuals to live longer will bias the financially weighted mortality experience of the portfolio. In particular, a lives-weighted measure like Equation 8.1 would be an under-estimate of the actual financial experience of a portfolio if longer-lived beneficiaries had larger benefits than average (this is exactly the case with longevity risk, as shown in Figure 8). Richards (2008b) and Richards and Currie (2009) show in detail how better-off pensioners have a longer life expectancy.

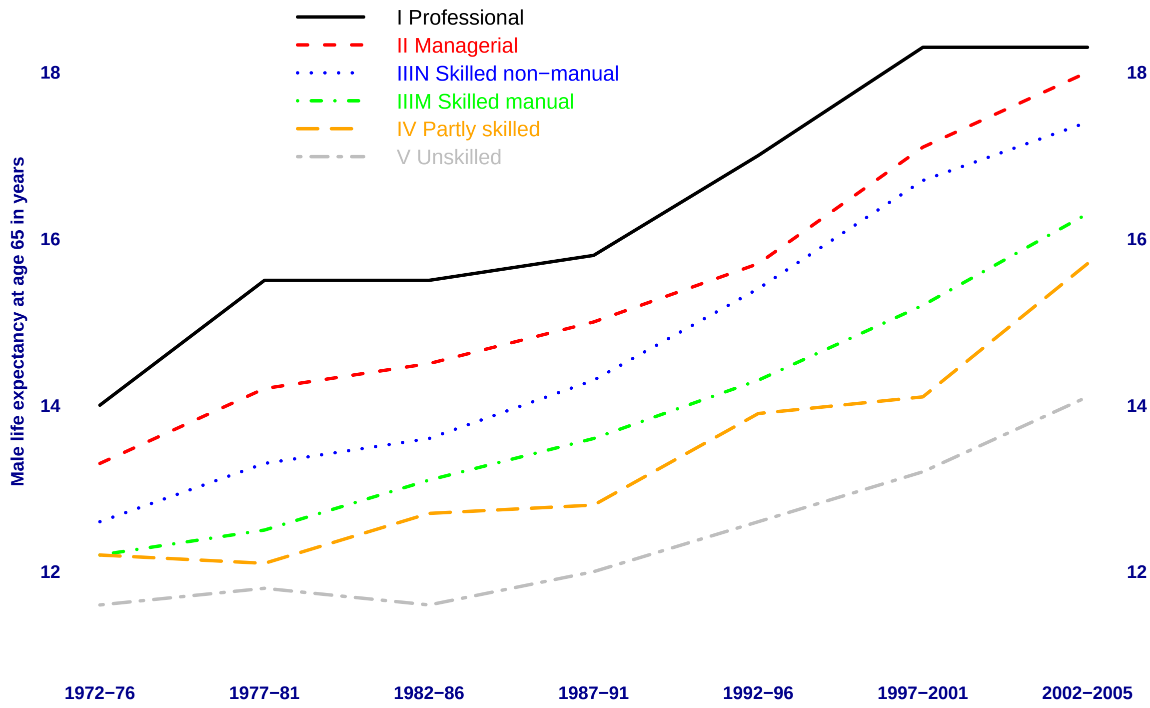

Figure 8.1: Trends in male period life expectancy at age 65 by socio-economic group. Source: ONS (2006).

In national statistics, socio-economic group is determined by occupation (ONS, 2002). However, such detailed information is seldom available for actuarial work, so the historical approach to allowing for such differentials has been to produce separate mortality tables on both a lives (or policies) basis and also with deaths and exposures weighted by pension size. Since people of higher socio-economic status tend to have larger pensions (Richards and Currie, 2009), this weighting by amounts can implicitly allow for socio-economic differentials in aggregate. This would be allowed for by modifying Equation 8.1 as follows:

$$\frac{\sum_{i=1}^{n} w_i d_i}{\sum_{i=1}^{n} w_i q_i} \tag{8.2}$$

where w_i is the measure of policy value or importance, which in the case of pension or annuity business is most commonly the annualised pension or reserve. Equation 8.1 would be referred to by actuaries as a lives-weighted calculation, whereas Equation 8.2 would be an amounts-weighted calculation. For example, on a lives basis CMI

(2008) gives the complete period expectation of life for male pensioners aged 65 as 16.7 years, but this rises to 18.1 years when using the equivalent table weighted by pension size.

Recent generations of standard tables have gone further than simple weighting by amounts and now also include separate tables by pension size-band (CMI, 2008). Further risk factors can be taken into account by producing separate comparisons, for example for those retiring at normal retirement age and those retiring earlier, often in poorer health and with subsequent elevated mortality (CMI, 1999). An improvement to the historical approach of comparing against standard tables is to build a statistical model or contingency table. This uses the same grouped data, but this time with an explicit probability distribution for the number of deaths observed in each sub-category. This allows formal tests of goodness of fit, which are otherwise trickier to achieve when comparing against a standard table as in Equations 8.1 and 8.2.

However, the problem with these approaches to mortality is that the repeated sub-division can quickly exhaust the credibility of even a large data set — each individual can only contribute to one of the sub-divisions. Data are seldom evenly spread across risk factors, so even a large data set might not be able to support the investigation of mortality for certain combinations of risk factors. This applies to any approach which involves sub-dividing the data set, including the empirical survival curves described by Kaplan and Meier (1958). Figure 8 shows the near-perfect smoothness when the data for a large portfolio is divided by gender. However, further sub-division leads to progressively less smooth survival curves, as shown in Figure 8.

A more efficient approach is therefore to build a statistical model for the risk factors at the level of the individual. This can be done using either q_x or μ_x, and the advantage is that each individual can contribute to the estimation of as many risk factors as (s)he possesses. Modelling mortality at the level of the individual means that there is no practical limit to the number of risk factors which can be investigated (Richards, 2008b).

There are other practical benefits from the switch to modelling mortality at the level of the individual. Historically, modelling of the mortality of groups had to contend with the problem of *over-dispersion*, i.e. the tendency for mortality counts to

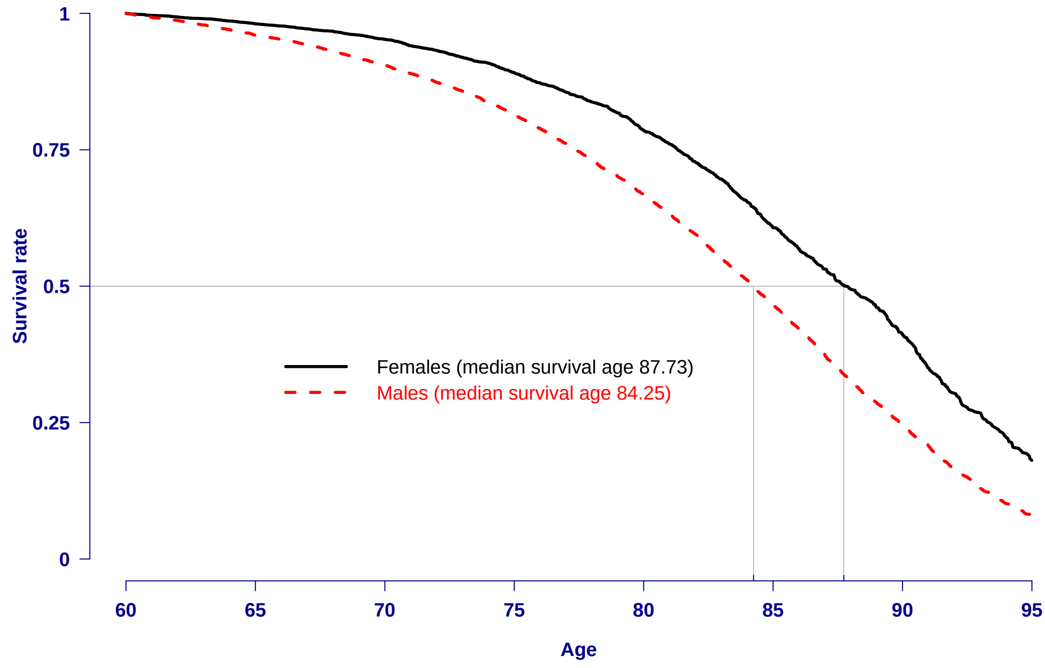

Figure 8.2: Kaplan-Meier survivor function by gender. Source: Richards (2010b).

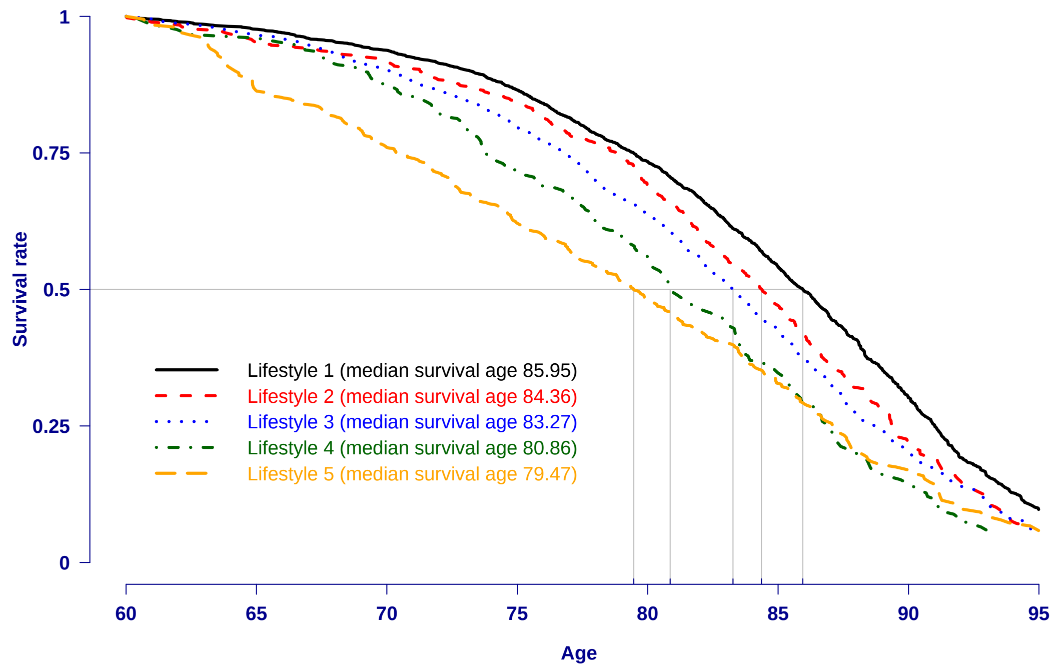

Figure 8.3: Kaplan-Meier survivor function by lifestyle group, where the groups are determined by the postcode mapping process described in Richards (2008b).

exhibit greater variability than allowed for in the standard probability distributions. Over-dispersion is a phenomenon often observed in population-sized data sets, for example where seasonal fluctuations in mortality mean variation around the trend is greater than would otherwise be expected. An illustration of season variation in mortality is given in Figure 8:

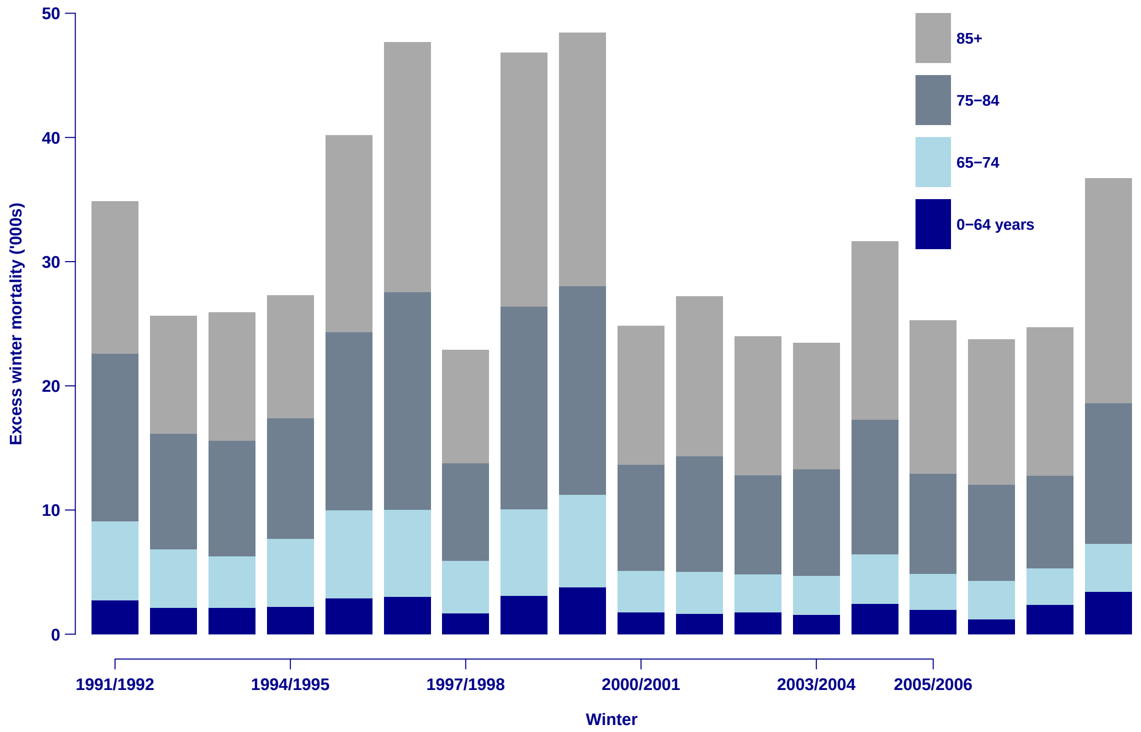

Figure 8.4: Excess winter deaths in England & Wales by age group. Source: Richards (2008a).

However, in the types of data sets typically encountered by actuaries, over-dispersion commonly arises because data is collected on policies, not people. People can do have more than one policy, and this tendency is strongly correlated with socio-economic status, as shown in Figure 8 (more detail is given in Richards and Currie (2009)). The problem of duplicates is therefore critical for mortality modelling, as Figure 8 shows that socio-economic status is both a key risk factor and strongly correlated with duplicates.

A classic example of the impact of duplicates was documented in the graduation of the UK standard table known as a(90) (CMI, 1976). The value for q_{94} was many times higher than expected, and this was found to be because:

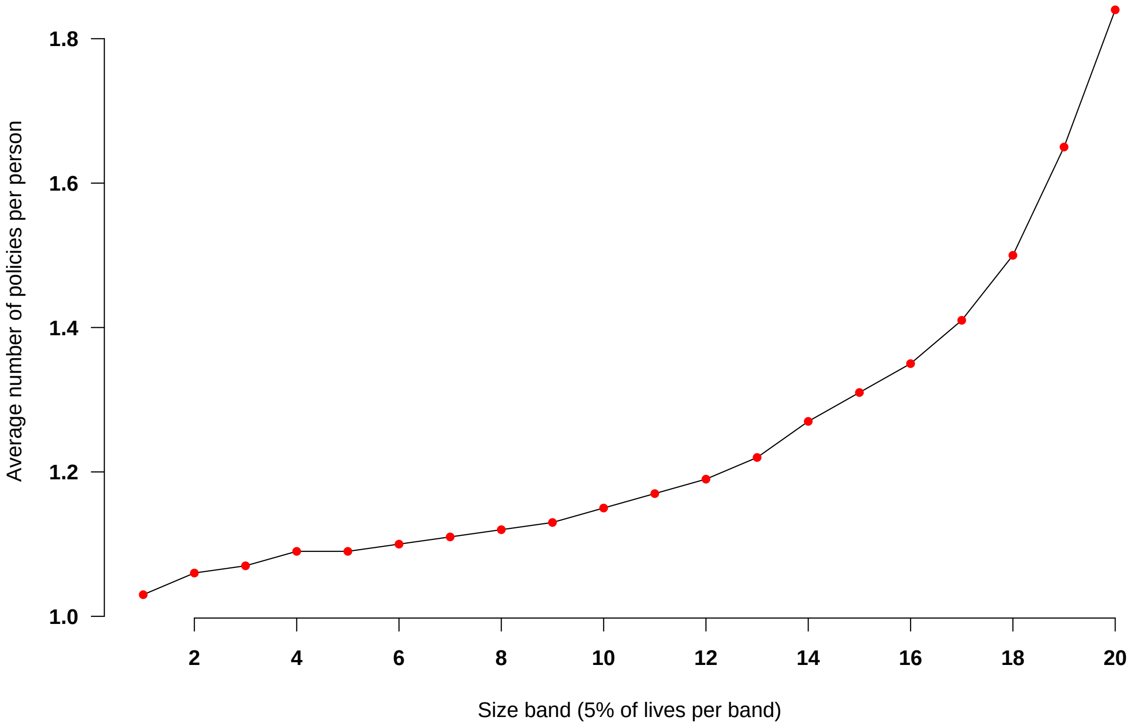

Figure 8.5: Average number of policies per person in each of equal-sized membership bands ordered by total annual annuity income. Band 1 is the 5% of lives with smallest annual pensions, through to band 20 which is the 5% of lives with the largest annual pensions. Source: Richards and Currie (2009).

> of 54 identified deaths in the combined pre-1957 and post-1956 data some 41 were of a Mr A and 3 of a Mr B, so there were only 12 separate lives for the 54 policies
>
> CMI (1976)

Actuarial literature therefore devoted considerable efforts to handling the over-dispersion which existed in count data, including Daw (1951) and Djeundje and Currie (2009). However, this treats one of the *symptoms* of over-dispersion, but not its root cause. Using individual-level data allows actuaries to deal with over-dispersion by eliminating the source of the problem: Richards (2008b) outlined a detailed process of *deduplication* to take policy-orientated data and turn them into data on independent lives for modelling. With duplicates eliminated during data preparation, there is less need to allow for the effects of over-dispersion caused by those duplicates. In addition to enabling efficient use of data in mortality modelling, using individual-level data can eliminate the problem of over-dispersion as well.

The move away from crude comparisons to standard tables and towards statistical models is led extra urgency by the judgment by the European Court of Justice (ECJ) banning the use of gender in insurance pricing with effect from December 2012. This is a major change for pricing longevity risk — Richards and Jones (2004) found that gender was the second most important risk factor for longevity after age. If insurers can no longer use this as a rating factor in pricing, then it becomes all the more important to make the best use of other available risk factors. An interesting aspect is the areas of insurance business which may not be affected by this. For example, reinsurance transactions look likely to be out of scope of the ECJ ruling, and some pension-scheme risk-transfer products look to be similarly unaffected. The original text of the 2004 Directive states:

> the use of sex as a factor in the calculation of premiums and benefits for the purposes of insurance and related financial services shall not result in differences in individuals' premiums and benefits.
>
> Article 5(1) of Council Directive 2004/113/EC of 13 December 2004

The key phrase here is "individuals' premiums and benefits". An insurance company is not an individual, and individual policyholders' premiums and benefits are not affected if liabilities are reinsured. Thus, it would appear that the entire business of reinsurance and retrocession in the European Union is out of scope of this judgement. Similarly, neither a company nor the company's pension scheme are individuals, and the benefits received by individual scheme members are unaltered by any risk-transfer solutions the scheme may put in place to protect itself. Thus, the markets for bulk annuities and longevity swaps should both be unaffected as well. Individual annuity contracts will be affected, which means the same risk will be priced differently depending on which market it appears in.

Note also that a difference will open up between the risk factors an insurer is allowed to use in pricing, and the risk factors an insurer should use for reserving. In the case of individual annuities, gender will be forbidden as an underwriting factor after 2012. However, since strong mortality differentials exist between males and females, gender will still have to be used as a risk factor for reserving and for internal modelling.

In addition to the move towards statistical models, modern actuaries are also increasingly looking to model the force of mortality, μ_x, instead of the probability of death, q_x (see Chapter 5 for definitions). One reason is the greater simplicity of modelling mortality in the presence of other decrements, i.e. where there are competing risks: when modelling μ_x, no further assumptions are required. In contrast, modelling q_x requires additional assumptions about the inter-relationship between the decrements which may not hold true in practice (Macdonald, 1996). In the case of longevity risk there typically are no competing risks, but insurers still find that continuous-time modelling is superior because it can make better use of the available data. One example of this is given in Table 8.1:

Table 8.1: Mortality data available for μ_x and q_x over 2004–2006. Source: Small life-office annuity portfolio.

	Data available for μ_x			Data available for q_x		
Age	Lives	Time lived	Deaths	Lives	Time lived	Deaths
60	4,804	3,528.5	32	4,054	3,185.6	31
61	4,572	3,440.9	39	4,388	3,065.4	38
62	4,285	3,040.9	33	4,087	2,635.6	33
63	3,802	2,731.9	48	3,679	2,671.9	48
64	3,660	2,668.2	44	3,544	2,614.5	44
65	5,822	4,336.6	47	5,225	4,051.2	44

As Table 8.1 shows, the requirement for a full year's exposure for a q_x model has reduced the data available. For example, the number of lives contributing exposure at between age 60 and 61 is reduced by 16%. A model for q_x could of course handle this by making some assumptions about the distribution of deaths over the year and adjusting the model. However, this complicates things unnecessarily, and it is all too easy to get the adjustment wrong, as demonstrated in the discussion of Madrigal et al (2009).

Three key aspects of modelling mortality are smoothing, interpolation and extension. Mortality statistics are subject to random fluctuations, so a mortality model should seek to smooth these to find the underlying mortality rates:

> graduation must smooth out irregularities due to random variation [...] while maintaining all the essential underlying variations in the mortality pattern.
>
> Heligman and Pollard (1980)

Historically, tables could be smoothed by using moving averages applied to mortality rates, but this suffers from two flaws: the first is that this can only give mortality rates at ages where there are data, and second it assigns an equal importance to each rate regardless of how much data lies behind it. The ability to extrapolate fitted mortality rates to ages where there are no data require the assumption of a *mortality law*, i.e. a functional form which mortality rates follow. The earliest example of this came from Gompertz (1825), who introduced the idea of a log-linear pattern in mortality, followed by many others: Makeham (1859), Perks (1932), Beard (1959) and Heligman & Pollard (1980).

Chapter 9

Projecting future mortality rates

The projection of future mortality rates is of considerably wider societal interest than the narrow confines of actuarial work. For this reason much work on mortality projections has taken place outside the actuarial profession. Indeed, the actuarial profession has very much been overtaken in this respect. By way of illustration, Lee and Carter (1992) presented a landmark stochastic projection model:

$$\log \mu_{x,y} = \alpha_x + \beta_x \kappa_y + \epsilon_y \tag{9.1}$$

where $\mu_{x,y}$ denotes the mortality hazard rate at age x in year y. This is not the parameterisation presented by Lee and Carter (1992), but it is the parameterisation used by Richards and Currie (2009, 2011). The parameter α_x represents the level of mortality at age x, while β_x is an age-related response to the time-based effect κ_y. The parameters ϵ_y are presumed to be identically distributed error terms with mean zero and constant variance, σ^2. The innovative aspect of the Lee-Carter model is that it reduces the two-dimensional problem of projecting by age and time into a one-dimensional problem of projecting κ_y. As a stochastic model, the Lee-Carter model provides not just a best-estimate projection, but can also give insight into the uncertainty surrounding that projection. The Lee-Carter model has been repeatedly adapted and extended: by Delwarde, Denuit and Eilers (2007), who smoothed the β_x parameters to reduce the likelihood of inconsistent forecasts at adjacent ages, and by Richards and Currie (2009) who smoothed both the β_x and κ_y parameters and thus replaced the usual time-series projection of κ_y with a penalty projection of Currie, Durban and Eilers (2004). The Lee-Carter model is not without its drawbacks, however — it is incapable of dealing with cohort-based mortality patterns, for example, and Kingdom (2008) illustrated that the implicit assumption of a constant set of β_x in the Lee-Carter model was questionable in some populations.

In contrast, even ten years after Lee and Carter's paper, the actuarial profession was still producing deterministic — and in some respects arbitrary or subjective — projection bases for future mortality improvements. Examples include CMI (2002) in the UK, but also DAV-Unterarbeitsgruppe Rentnersterblichkeit (2005). A hybrid approach was used in Italy where the actuarial tables incorporated allowances for improvements based on a population projection, but where the population projection itself was based on a variant of the Lee-Carter model (Cocevar, 2007). As recently as CMI (2010) the UK actuarial profession produced a projection model which was not only deterministic, but also contained no fewer than 1,048 separately modifiable parameters.

There are a number of ways of sub-dividing methodologies: (i) deterministic v. stochastic, (ii) extrapolation v. expectation or (iii) all-cause v. cause-of-death. For modern work the key limitation of a deterministic projection is that it is certain to be wrong. Stochastic models are therefore preferred because they acknowledge the inherent uncertainty in projecting into the future. Booth and Tickle (2009) give a wide-ranging overview of the various methods of mortality projection, in which they also contrast extrapolation (continuing a recent trend in the data) with expectation (where future mortality approaches some limiting value). A key limitation of expectations is that they, too, are often wrong:

> [e]xpectation is not generally a good basis for mortality forecasting, as it is subjective; expert expectations are invariably conservative
>
> Booth and Tickle (2009)

This latter point was hammered home by Oeppen and Vaupel (2002), when they observed that between 1928 and 1990 the predicted limits to life expectancy were broken "on average 5 years after publication" of the supposed limit. Booth and Tickle (2008) mention an interesting feature of expectations called "assumption drag", i.e. where expert expectations often prove to lag actual experience, rather than lead it as might be hoped for. Another problem is the phenomenon of "expert flocking", which arises from a common information base and individuals' reluctance to stray too far from consensus.

Richards (2010c) described a string of practical problems with using data disaggregated by cause of death as a foundation for mortality projections. These were illustrated with specific examples for the data available for England and Wales, but the problems apply to most nations' data. In advancing a methodology which could apply to cause-of-death data, Oeppen (2008) noted a crucial lack of detail in practice:

> deaths are often tabulated by 5 year age groups and the open age interval into which the deaths of the oldest-old are aggregated is often defined at a relatively young age such as 85. Unfortunately, it is at these high ages where most of the temporal dynamics are occurring
>
> Oeppen (2008)

Since age is almost always the most important risk factor in mortality work, this is a critical practical shortcoming on top of the theoretical objections to cause-of-death projections. For actuarial use, therefore, the most useful projection models are:

- Statistical, in that they are parameterised by data,
- Stochastic, in that they acknowledge uncertainty,
- Extrapolative, in that they avoid subjective beliefs, targets or limits, and
- Based on all-cause mortality data to avoid the problems with disaggregated cause-of-death statistics.

One illustration of model risk is how a new insight can change the direction and basis of modelling and thinking. Willets (1999) brought the so-called "cohort effect" to the attention of the UK actuarial profession, followed up by Willets (2004) and Willets *et al* (2004). The cohort effect could in fact be described as having been re-discovered by the actuarial profession, as year-of-birth patterns had already been noted by Derrick (1927). Willets (1999) triggered a flurry of revisions to mortality projections in the UK actuarial community — for example, the deterministic, period-based projections of CMI (1999, 2000) were replaced by cohort-based projections of CMI (2002). The need to acknowledge uncertainty over future projections led to CMI (2005, 2007), although an unwelcome lapse back into deterministic, expectation-based models has recently occurred with CMI (2010).

Chapter 10

Contrasts in modelling current and future mortality

There are good reasons for actuaries to separate mortality modelling into the two components of (i) current rates and differentials, and (ii) projecting future mortality rates. The most obvious reason is data: a given portfolio will often have individual-level mortality data which allows the building of a mortality model of individual risk. In contrast, the long time series required for projections are typically only available with grouped data with minimal sub-division by risk factors such as age and gender.

However, there are also qualitative reasons for separating the modelling of mortality differentials and performing projections. When modelling mortality differentials, goodness of fit between model and data is key — the whole point of building a model of differentials is to accurately identify the strength and timing of any variation in mortality rates between groups, usually for accurate pricing or reserving. In contrast, the goodness of fit between model and data is not necessarily a primary consideration for a projection model. Booth and Tickle (2009) wrote that:

> in-sample errors are not necessarily a good guide to forecast errors. For long-term forecasting in particular, the choice between models cannot reliably be based on historical goodness of fit
>
> Booth and Tickle (2009)

This leads to a paradox: a model which fits the data well may be a poor projec-

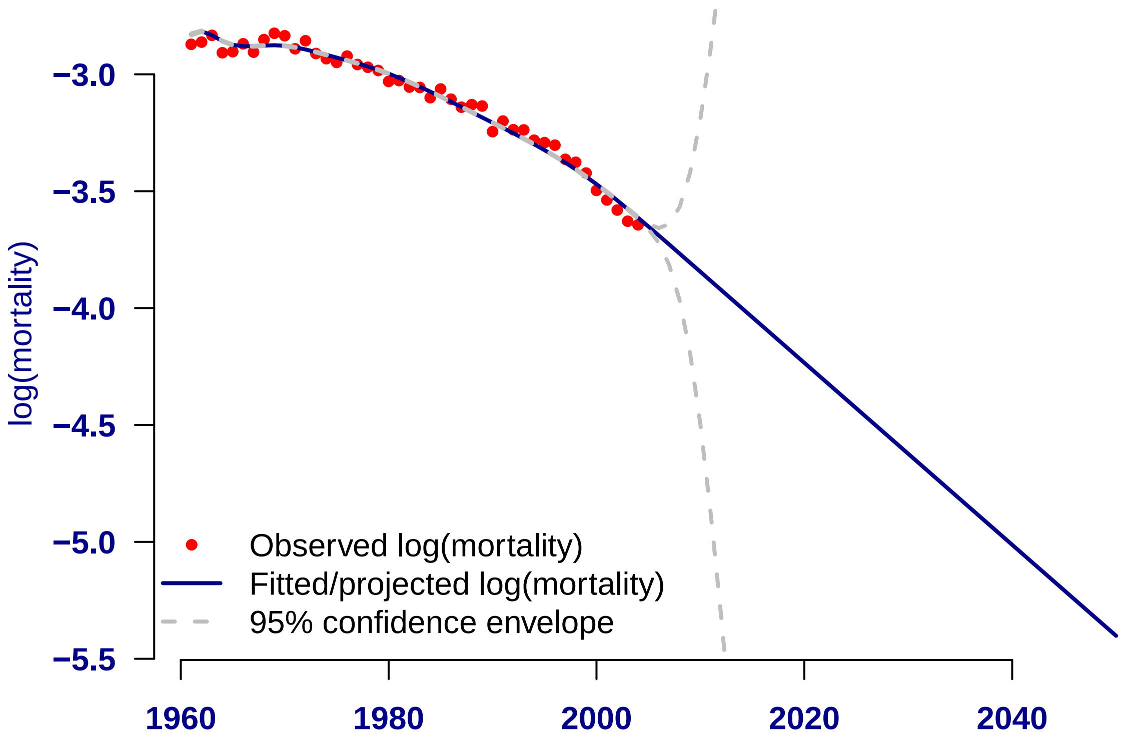

Figure 10.1: Central projection and confidence intervals for Currie-Richards (2009) model applied to mortality of males in England and Wales between ages 40–104 over 1961-2007. To find the optimal values of the smoothing parameters for age and time, λ_x and λ_y, respectively, the BIC is minimised, resulting in $\lambda_x = 619.7$ and $\lambda_y = 1.767$. However, although this combination of smoothing parameters produces the lowest BIC, it leads to drastic undersmoothing in the time direction with corresponding catastrophic consequences for the confidence envelope.

tion model, while a model which fits the data poorly may nevertheless be a perfectly serviceable projection model. In modelling mortality differentials, however, a model which does not fit the data well is simply a poor regression model. This has consequences for model choice: in modelling mortality differentials, we use an information criterion such as the AIC (Akaike, 1987), BIC or DIC to select a model or the risk factors therein, or perhaps also the deviance — see McCullagh and Nelder (1989) and Collett (2003). For projection models, targetting the optimal information criterion can produce a poor choice, as illustrated in Figure 10.

The problem illustrated in Figure 10 is not limited to the model in Richards and Currie (2009) and has been observed in other penalty-based models — see Richards *et al* (2009). Nor is the problem necessarily restricted to projection models based on a penalty function. In selecting an ARIMA(p, d, q) model for projecting κ_y in Equation

9.1 the most obvious solution to choosing p, d and q is to pick those values which best fit the observed behaviour of κ_y, say by minimising an information criterion such as the AIC. In the author's experience the use of the AIC to select p, d and q has never yet selected an obviously poor projection model, but this does not seem guaranteed. Figure 10 shows that some combinations of p, d and q can result in very poor forecast behaviour, and it is plausible that there will be data sets whereby these poor-performing choices will just happen to have the lowest AIC (or other information criterion).

Since goodness of fit is not a reliable criterion for selecting a projection model, Cairns *et al* (2009) developed a series of criteria for judging the projection outputs themselves. Nevertheless, a choice of projection model invariably involves more subjectivity than the choice of a model for mortality differentials.

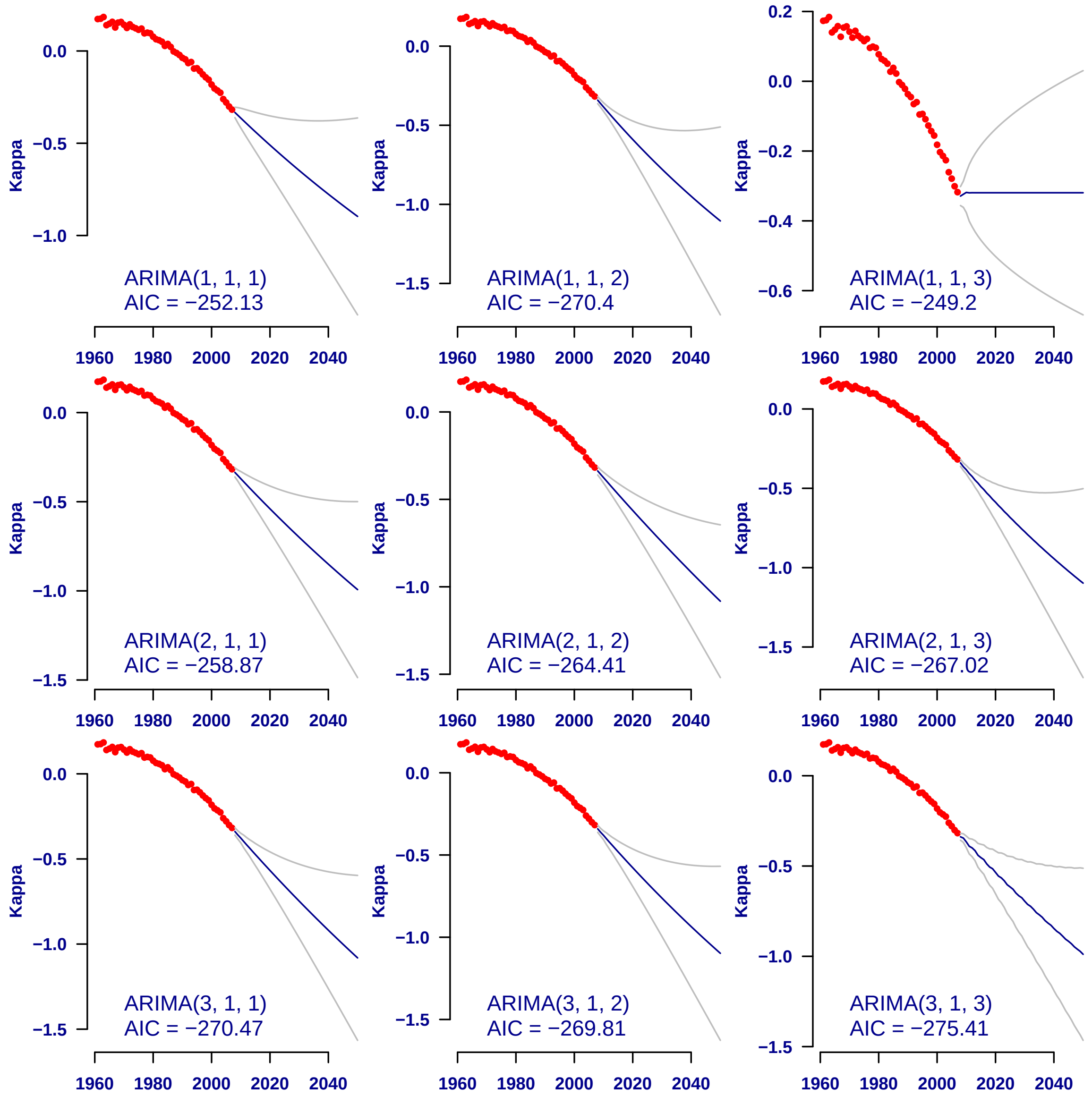

Figure 10.2: Central projection and confidence intervals for ARIMA(p, 1, q) projections of κ_y for the model of Delwarde, Denuit and Eilers (2007) applied to mortality of males in England and Wales between ages 40–104 over 1961–2007. The ARIMA(1,1,3) model in the top right is clearly not sensible for projections, and it happens to have a poorer AIC value than the other combinations. However, this does not seem guaranteed and it is possible that there will be data sets where the best-fitting ARIMA(p, d, q) model happens to have unacceptable behaviour.

Chapter 11

Model risk

When modelling actual mortality experience of a portfolio, model risk is seldom an issue. Data are usually relatively plentiful, often covering thousands of deaths in mid-sized insurance portfolios. Alternative models for mortality can be fitted and their goodness of fit assessed by an information criterion such as Akaike's Information Criterion or AIC (Akaike, 1987) or the Bayesian Information Criterion. The "correct" model for such a regression problem can never be known, but for most portfolios it is typically possible to find a close-fitting model which passes all goodness of fit tests. With the resulting best-fit model, the broad shape of the fitted mortality rates will be sufficiently close for the risk of the wrong model to be manageable in most cases.

For projections, however, model risk is as big a source of uncertainty as, say, the 95% confidence intervals produced by a model itself. Richards and Currie (2009) demonstrated this by using the same data set and two slight variations in the parameterisation of a Lee-Carter model. As can be seen in Figure 11, the model risk is as important as the uncertainty over the direction of future mortality within a given model.

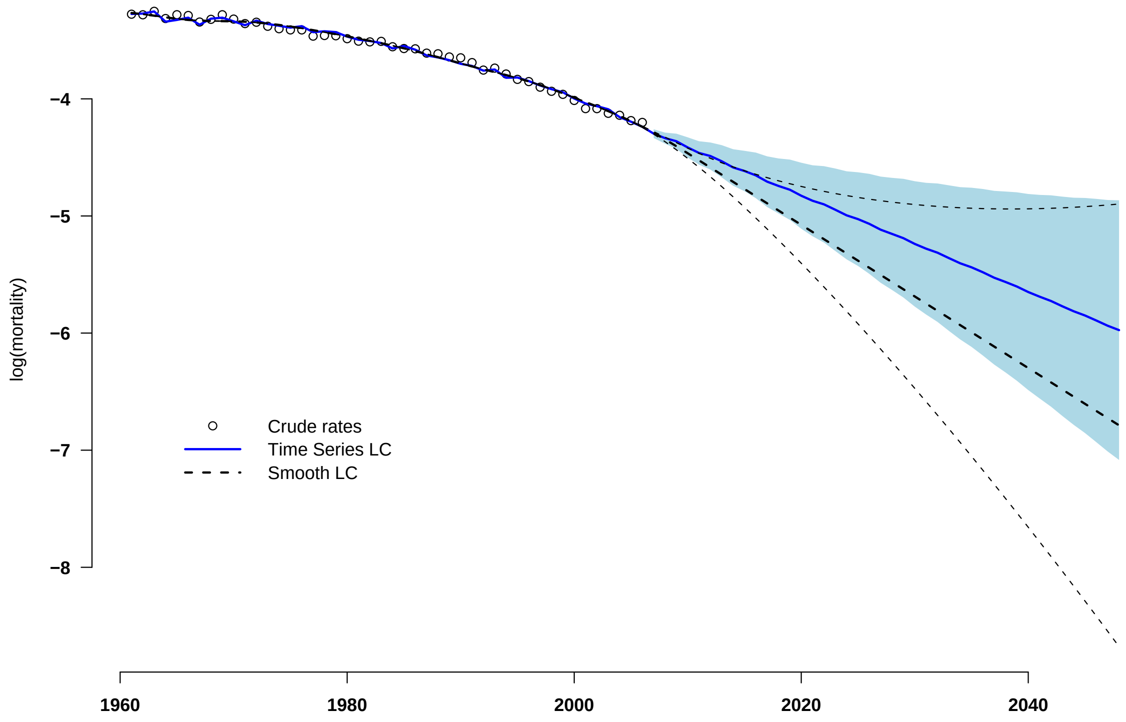

Figure 11.1: log(mortality) at age 65 for males in England & Wales with differing projections and confidence intervals from two minor variants of the Lee-Carter model. Source: Richards and Currie (2009).

Chapter 12

Basis risk

In actuarial work, basis risk is the name given to the risk arising from applying a model based on one population to the assessment of a different population. Basis risk does not normally apply to the modelling of current mortality rates, since the rates derived from modelling the portfolio can then be applied to valuing and understanding the same portfolio. A degree of basis risk can arise if the fitted model is not sophisticated enough, however. For example, many pension schemes in the UK experienced a shift from blue-collar to white-collar workers. If a mortality model did not include this status as a risk factor, then there would be a form of basis risk in using that model even though it was calibrated to the same portfolio. This risk is not new and applies wherever a mortality model does not include all relevant risk factors:

> The results now point very strongly to the likelihood that since the Finance Act 1956 there has been a change in the class of life purchasing annuities
>
> CMI (1966)

In the example cited above, mortality data was collected and analysed by age, gender and policy size, which proved to be insufficient when the mix of lives changed due to an unobserved risk factor — the "class of life"according to the CMI. Applying the mortality rates from the pre-Finance Act business to the post-Finance Act business was therefore inappropriate — an example of basis risk. Basis risk can be controlled to a considerable extent by adding risk factors to the mortality model, as done by Richards and Jones (2004), Richards (2008b) and Madrigal et al (2009). The more valid and transferable risk factors are included in a model, the less scope there is for differences to arise due to an unobserved risk factor.

In contrast, basis risk is almost a given for mortality projections. Models such as those from Lee and Carter (1992), Cairns, Blake and Dowd (2006) and many others are based around having a reasonably long history of mortality data, say for at least twenty years. Few portfolios have this kind of historical information of their own. This is often due to changes in administration — pension-scheme administration is often outsourced, for example, and when administrators are changed the mortality history data is usually lost. Within life insurers, migration from one computer administration system to another is also often accompanied by the loss of historical mortality data. As a result, most projection models are parameterised using an unrelated data set which happens to have the required historical data. Often this is population data, of which recipients of private pensions (for example) will be a small and very select sub-group of wider society. This difference gives rise to basis risk in mortality projections.

Basis risk in projections is so common that actuaries often forget that it is there. However, annuitants and pensioners are typically drawn from a select sub-group of the wider population, so differential rates of improvement are a risk when a projection model is calibrated using population data. For example, Figure 8 shows not just the extent of socio-economic differences in life expectancy, but also that these differentials have widened from around two years in 1972–76 to four years in 2002–05. Improvements for socio-economic groups IV and V appear to have been less strong than for the other groups. In practice, groups IV and V seldom appear in portfolios of private annuities or pensions, and so calculations of population-based mortality improvements are likely to under-state the improvements actually experienced in portfolios analysed by actuaries.

Chapter 13

Parameter risk

With any statistical model there is some degree of uncertainty over the estimated parameter values. When modelling portfolio mortality rates, standard errors are often relatively small compared to their corresponding estimates. This is illustrated in Table 13.1, which shows the coefficient of variation (the ratio of the standard error of an estimate to the estimate itself — see McCullagh and Nelder (1989)).

Table 13.1: Parameter estimates for Perks model of mortality for large portfolio of life-office pensioners aged between 60 and 95 over the period 2000–2006. Source: Richards (2009).

Parameter	Estimate	Standard error	Coefficient of variation
Age	0.115076	0.0008	0.7%
Gender.F	-1.75126	0.1090	6.2%
Gender.F:Age	0.0165734	0.0014	8.4%
Intercept	-11.8655	0.0593	0.5%
Time	-0.0366379	0.0024	6.6%

Parameter uncertainty in projection models is usually larger by comparison, as shown in Table 13.2. The coefficients of variation are generally larger, despite the ARIMA model being calibrated to a much larger data set observed over a much longer period of time — the population data behind Table 13.2 has 255 times more deaths than the life-office data behind Table 13.1. This uncertainty over the parameters expresses itself in a widening funnel of doubt as a projection moves forward in time.

Parameter risk is relatively modest in analysing mortality differentials, although Table 13.1 shows that it is relatively larger for time-varying factors than for main

Table 13.2: Parameter estimates for ARIMA(3,1,3) model for κ_t in Lee-Carter (1992) model for males aged 50–104 in England and Wales over the period 1961–2007. “ar” denotes an autoregressive parameter and “ma” denotes a moving-average parameter.

Parameter	Estimate	Standard error	Coefficient of variation
ar1	0.362	0.061	16.9%
ar2	-0.304	0.062	20.4%
ar3	0.928	0.051	5.5%
ma1	-0.584	0.113	19.3%
ma2	0.792	0.126	15.9%
ma3	-0.834	0.112	13.4%

effects. In contrast, parameter risk is one of the defining features of the expanding funnel of doubt for mortality projections — see Figure 11 — and is therefore a key part of the cost of uncertainty for reserving for pensions and annuities. Furthermore, increasing the volume of data for a portfolio will reduce parameter uncertainty in a model of mortality differentials. In contrast, more data may not make a material difference to the parameter uncertainty in a projection model.

In a sense the above comparison is unfair: in Table 13.1 there are over 350,000 lives contributing to the estimate of the time-trend parameter, for example. By way of contrast, the ARIMA process for κ_t in the Lee-Carter model in Table 13.2 is calibrated using separate observations for just 47 years. In practice, however, gathering a much longer series of historical information raises the question of relevance. In a critique and extension of the model by Cairns, Blake and Dowd (2006), Sweeting (2011) used data for England & Wales starting in 1840. Even if the data that far back in time were reliable, it is questionable that they are relevant to modern times. The broad conclusion therefore still stands: measurement of current mortality differentials should always improve in accuracy with more data, while the problem of projection uncertainty will always remain.

Chapter 14

Idiosyncratic risk

The future lifetime of a pensioner is an unknown quantity — even if we assume that the model for mortality is precisely known, there is still uncertainty over the length of an individual's future lifetime. This is idiosyncratic risk — an individual's future lifetime is a random variable[1].

As a result of the law of large numbers, the impact of idiosyncratic risk reduces as the number of pensioners in a scheme grows. Richards and Currie (2009) illustrate the impact of idiosyncratic risk by looking at 10,000 simulations for three portfolios: (i) a pension scheme of 2,268 lives, (ii) a small annuity portfolio of 15,429 lives, and (iii) a large annuity portfolio of 207,190 lives. In each case the same fixed underlying model is used for projecting mortality and the only random element was the simulation of the individual lifetimes. In each case the 99.5% most expensive discounted cost of providing a benefit of £1 per annum was expressed relative to the median discounted cost. For the pension scheme the 99.5% most extreme cost was 2.02% higher than the median, while for the small annuity portfolio it was 0.63% and for the large annuity portfolio it was just 0.20%. This illustrates that portfolios with larger numbers of lives face less idiosyncratic risk.

An important related aspect is concentration risk. Pension schemes and annuity portfolios are inherently unequal places, as shown in Richards (2008b) and Richards and Currie (2009). One measure of inequality of income is the Gini coefficient (Gini, 1921), which takes values between 0 (perfect equality) and 100 (all income received

[1]This is the essence of survival models: a model for the force of mortality is directly equivalent to saying the future lifetime of an individual is a continuously distributed random variable (and vice versa).

by one person). For example, the United Nations Development Programme (2010) reported an Income Gini coefficient for the UK of 36.0 for the period 2000–2010. In contrast, the large annuity portfolio examined in Richards (2008b) had a Gini coefficient of 66.0, meaning that this annuity portfolio was substantially less equal than society as a whole. This is not unusual for private pensions in the UK — the large portfolio of defined-benefit pensions analysed in Richards (2008b) had a Gini coefficient of 60.9. The data for the two portfolios covered 2000–2006, making the portfolios' Gini coefficients broadly contemporaneous with the UN-calculated one, and thus illustrate that private-sector pension-scheme benefits and annuity payments are more unequally distributed than income in society in general.

An alternative to the Gini coefficient is to show the proportion of liabilities for each decile of lives. An example of this is shown in Figure 14 for two portfolios, one of life-office annuities and the other of defined-benefit pensions.

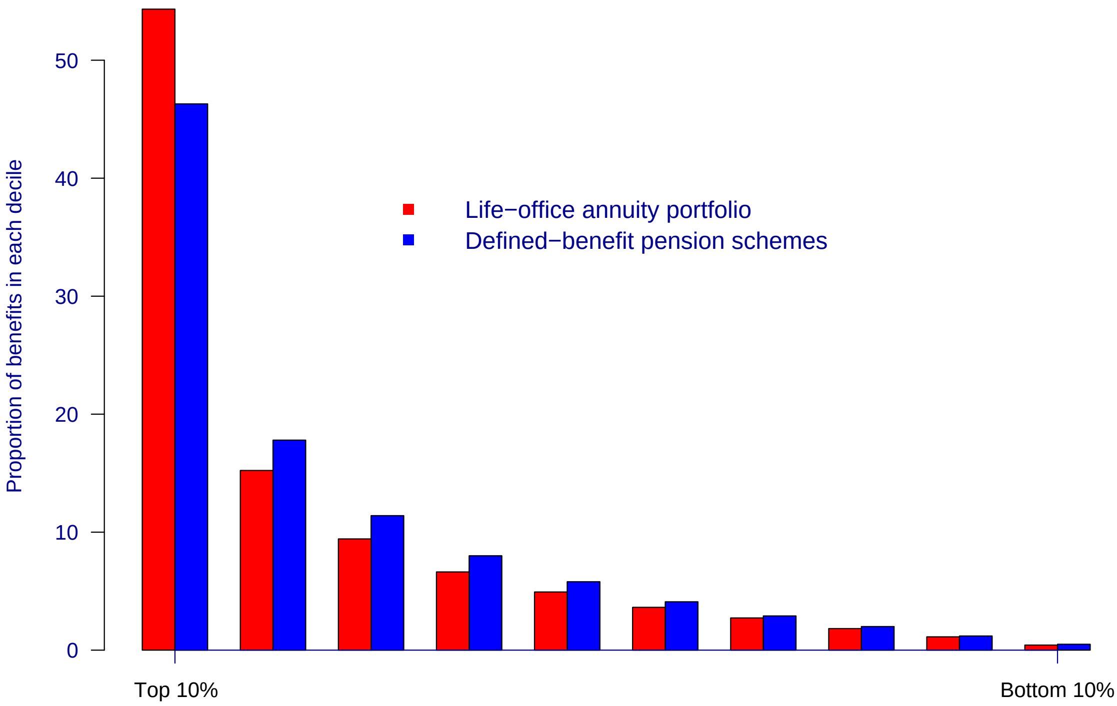

Figure 14.1: Concentration of benefits by decile for a life-office annuity portfolio and a collection of defined-benefit pensions. Source: Richards (2008b).

Figure 14 shows that pension benefits are heavily concentrated, with the top 10% of lives receiving around half of all benefits. This concentration of liabilities carries an extra risk, namely that the lives with the largest proportion of liabilities have a markedly different life expectancy than the ones with the smallest proportion. This is highly likely in most situations, as shown in Figure 8 and demonstrated in detail in Richards and Currie (2009).

Leaving aside any differences in life expectancy, one consequence of concentration risk is that it increases the volatility of financial measures of mortality, and operates against the law of large numbers. In practice this means that the financially weighted mortality experience of a portfolio will exhibit a greater volatility than would be expected than if benefits were distributed equally. Richards and Currie (2009) performed the same run-off simulations mentioned above using the actual pension benefits paid. In each case the 99.5% percentile for the discounted cost of benefits was higher: for the pension scheme the extra cost rose from 2.02% to 4.52%, for the small annuity portfolio it rose from 0.63% to 1.07% and for the large annuity portfolio it rose from 0.20% to 0.50%. In essence, the practical consequence of the inequality of pension benefits is to raise the run-off risk to the scheme and make its financially-weighted mortality experience behave like a smaller portfolio.

An interesting corollary of this is that actuaries cannot resort to the historical approach of so-called *model points* to calculate the impact of idiosyncratic risk. This was an approach developed in the early days of more modest computing power, where a small number of policies were selected such that their behaviour in deterministic calculations was broadly similar to the behaviour of the portfolio as a whole. Since idiosyncratic risk and concentration risk are specifically about the particular number of lives and their particular benefits, meaningful assessment of these two risks can only come from full-portfolio run-off simulation.

Chapter 15

Contributions to the papers in the accompanying thesis

This critical review accompanies the thesis made up of six papers published in peer-reviewed journals. Each of the six papers is listed below, together with the contribution made by each of the authors. The original content of each paper is highlighted.

Richards, Kirkby and Currie (2006) is a paper about the smoothing of two-dimensional mortality data and the insight gained into mortality patterns by year of birth. Stephen Richards was the editing author, and wrote sections 1, 2, 3, 4, part of 6 and 7. Dr Iain Currie wrote sections 5 and part of 6. James Kirkby wrote the software to fit the various models and generated the data for charting.

Richards et al (2007) is a paper about understanding past sources of mortality improvement and the limitations of various methods for projecting mortality rates into the future. Cause-of-death data were shown to be a useful means of understanding past improvements, but an imperfect means of explaining the mortality reductions implicit in a given projection scenario. Past mortality patterns were investigated for seven developed nations, with an emphasis on comparing age-period versus age-cohort mortality patterns. Some countries had pronounced cohort patterns, such as both genders in Germany and England and Wales, males only in France and Japan and females only in Sweden. However, period patterns in mortality appeared to dominate in other examples, such as both genders in the USA and females in France. Short demographic histories were given for each country to allow the reader to judge the extent to which age-period or age-cohort mortality patterns were explained by social history. Stephen Richards was the editing author, and wrote sections 1, 2, part of 4, 6, 11, 15, 17, 18 and 19. Text written by Dr Iain Currie from an earlier paper was reworked into section 10 by Stephen Richards with Dr Currie's permission. Joseph Lu co-wrote section 4 and wrote sections 12, 13 and 14. Jennifer Hubbard wrote section 5. John Ellam wrote section 3. Stephen Makin wrote sections 8, 9 and 16. Keith Miller wrote section 7.

Richards (2008a) is a paper about the extent to which cohort mortality patterns can be detected using limited data. The paper shows the limitations of the population estimates available for England and Wales: populations are estimated from census data at ten-year intervals, while deaths data are collected more or less continuously. Detailed investigation of the validity of the exposure data is carried out, showing that there are some material reservations about the quality of the population estimates which do not apply to the deaths data. A variety of methods are explored to assess the relative strength of time trends and cohort patterns, concluding that cohort patterns explain more mortality variation than a single constant time trend can. This is a single-author paper by Stephen Richards.

Richards (2008b) is a paper showing how the records of a life-company annuity portfolio, or a self-administered pension scheme, are ideal for the application of survival models. This is the first known published application of so-called geodemographic methods in actuarial work as a means of determining socio-economic differentials in longevity. This insight is not original, however, as the first known application of geodemographics to actuarial work was carried out by Richard Willets and Lawrence Andrews at the Prudential Assurance Company but never published. The validity of using geodemographics was confirmed subsequently by an independent paper by Madrigal et al (2009). This is a single-author paper by Stephen Richards, with programming for deduplication carried out by Gavin Ritchie.

Richards and Currie (2009) is a paper outlining alternative parameterization within a common Lee-Carter framework, and how the application of different models can lead to very different projections of future mortality rates. The impact of this model uncertainty is illustrated with reference to annuity reserves, and the resulting model risk is shown to be as important financially as the uncertainty over the projection within a given model. Stephen Richards was the editing author, and wrote sections 1, 2, some of 5, 6, 7, 8 and 9. He produced the graphs and wrote the software to perform the run-off simulations and valuations. Dr Iain Currie wrote sections 3, 4 and most of 5. He also wrote the software to fit the models used in the paper.

Richards (2010b) is a paper presenting a common parameterisation framework for survival models for actuarial use. The paper compares seven models from an actuarial background with ten models drawn from statistical literature. The basic features of the various models are compared, and the conclusion drawn is that the "actuarial" models provide a better fit as they seek to explicitly model the pattern of mortality by age. This is a single-author paper by Stephen Richards.

Chapter 16

Peer review

In most academic work peer review refers to the system of anonymous scrutineers reading and judging the quality of papers submitted to a journal. Further scrutiny takes place once the paper is published, as other authors are then able to read, comment and criticize further. In the UK actuarial profession there is a further element to peer review for papers published in the British Actuarial Journal, namely presentation of the paper to an audience of practising actuaries. Regular sessional meetings are held by the Institute of Actuaries and the Faculty of Actuaries (now merged as the Institute and Faculty of Actuaries). Papers to be discussed are made available to all UK Fellows and Students before a sessional meeting, and interested parties attend to debate the paper. Debates are not restricted to members of the actuarial profession, and for mortality-related topics there are often visitors from other disciplines. The discussion is then published alongside the paper itself in the British Actuarial Journal. Those unable to be present at the sessional meeting can and do make written contributions to the debate.

Four of the papers here have been presented to sessional meetings and have therefore undergone the extra scrutiny that this provides: Richards, Kirkby and Currie (2006), Richards et al (2007), Richards (2008b), and Richards and Currie (2009). All four of these papers are published with transcripts of the accompanying discussion. The papers covered by this thesis concern the application of modelling theory to practical problems in business, so it is important that such work is openly debated by an audience of practising actuaries. Every member of the UK actuarial profession has free access to the British Actuarial Journal, which makes it a natural vehicle for applied research papers such as those forming this thesis accompanying this critical review.

Chapter 17

Academic impact

One measure of academic impact is what other authors publish in relation to it, either building on one's own work or else commenting on it. This section describes some references to the author's published academic work in both the "building" and "commenting" categories.

The Continuous Mortality Investigation Bureau (CMI) is a UK organisation which gathers mortality and other statistics from member life offices and publishes reports. In 2005 the CMI released a tool for mortality projections which used the R scripts created to fit the models described in Richards, Kirkby and Currie (2006). The software integration work was carried out by James Kirkby.

Richards et al (2007) applied two-dimensional P-spline models to the mortality data of seven developed countries with widely varying social histories in the twentieth century. The aim was to find out which countries had year-of-birth patterns which were stronger than year-of-observation patterns, i.e. where cohort effects might be stronger than period effects. Some of the software source code used in preparing the paper was made available online[1] and this was used by Piero Cocevar to research mortality patterns for Italy. Cocevar (2007) shows that Italian males have cohort mortality patterns which are almost indistinguishable from those of males in England and Wales.

Richards (2008a) was attacked by Murphy (2010) for using the phrase "cohort effect" with the implication of a causal link between year of birth and subsequent mortality patterns. The points raised in Murphy (2010) were rebutted in detail in

[1] See http://www.richardsconsulting.co.uk/international.html

Richards (2010a).

Richards (2008b) demonstrated the use of geodemographic profiles in modelling mortality differentials of pensioners and annuitants. Madrigal et al (2009) followed up on this work and confirmed the importance of geodemographics using a different data set and a different geodemographic profiling system.

Chapter 18

Professional impact

Successful papers often lead to invitations to write summary articles in the actuarial press. For example, a direct result of Richards et al (2007a) was an invitation to write the article Richards (2007b). Similarly, Richards (2008b) led to an invitation to write the article in Richards (2008c), while Richards and Currie (2009) led to the invitation to write Richards (2009). Richards (2011) was written by invitation following Richards (2010c).

Other longevity-related articles which do not have direct links to papers in the thesis include Richards (2004), Richards and Robinson (2005), Richards (2007c) and Richards and Currie (2011).

The Faculty of Actuaries (now absorbed into the Institute and Faculty of Actuaries) awarded Richards et al (2007) the Faculty prize for best paper presented to a sessional meeting in the session 2006/2007.

As a result of the work in Richards et al (2007) and elsewhere, the author was invited to make available a free-to-use R script for actuaries to use when graduating mortality tables. This software can be downloaded at http://www.longevitas.co.uk/graduate.

Chapter 19

Commercial impact

Much of the author's research has been driven by requirements of clients in the management of longevity risk in pension schemes and annuity portfolios. The author has been a consultant on longevity risk since 2005, and the topics of research have been driven by the challenges facing clients. The first of these challenges is the search for risk factors for better underwriting and pricing, which led directly to the survival models in Richards (2008b, 2010b) and the application of postcodes for assessing socio-economic status in Richards (2008b). Another major challenge was regulatory changes for insurers: both the ICA regime in the United Kingdom and the forthcoming Solvency II regime in the European Union place emphasis on stochastic modelling. A key risk for writers of annuities lies in the uncertainty over future mortality improvements, which led to the search for new models — Richards, Kirkby and Currie (2006) and Richards and Currie (2009).

In addition to consulting on the modelling of mortality and longevity, the author runs three software businesses focused on the analysis of longevity risk. A key feature of these businesses is turning academic research into software tools which can be used in the practical management of longevity risk in company pension funds and life-office annuity portfolios. Many of the results published in the six journal papers in the thesis have been implemented in software subsequently licensed by insurers, reinsurers, investment banks and consulting actuaries. Some of the commercial applications of this research are described here in this section.

The author's main business is the Longevitas survival-modelling software suite. This allows actuaries to upload a simple extract from an administration database and validate and deduplicate the data for fitting survival models. Richards (2008b) describes in detail the entire process from start to finish, from data extraction and validation, through to deduplication and model-fitting and evaluation. Richards (2010b) details how the survival models are parameterized in a consistent framework.

The use of geodemographic profiles is not original in UK actuarial work, nor is the fitting of survival models to portfolio mortality data. However, both are tricky and the phenomenon of left truncation strictly limited what actuaries could do with standard statistical software. The Longevitas survival-modelling system made both very straightforward for actuaries, and the existence of detailed published methodologies in Richards (2008b) and Richards (2010b) helped in encouraging actuaries to use advanced statistical modelling in their daily work. As of February 2012 there were numerous companies using this software, and the survival-modelling techniques contained therein, including insurers, investment banks, reinsurers and a global consulting actuary. Most users are based in the United Kingdom, with a smaller number of licences in France, Germany and the USA.

The second software business is the service at www.mortalityrating.com, which provides small- and medium-sized pension schemes with an assessment of the longevity risk in their pensioner population. The automatically generated reports provide a rating based on five risk factors — age, gender, pension size, UK postcode and whether the pensioner retired early or not — using the sort of postcode-driven socio-economic model described in Richards (2008b). As part of each report, the pension scheme is also assessed for concentration risk, idiosyncratic risk and trend risk, including illustration of model risk as in Richards and Currie (2009). As of September 2011 mortalityrating.com was in use at several consulting organizations, mainly actuaries consulting to pension schemes.

The most recent software product offered by the author's business is the Projections Toolkit. This is a joint venture with Heriot-Watt University and the main contact person for this joint venture is Dr Iain Currie (co-author of two of the papers in the thesis and also co-supervisor for this thesis). The Projections Toolkit makes it easy for actuaries working in life offices and elsewhere to quickly fit stochastic projection models for use in ICA submissions and Solvency II work. Statistical projection models are not particularly straightforward to fit, yet the ICA requirements in the

UK (and the pending requirements of Solvency II throughout the European Union) mean that they are increasingly required. The Projections Toolkit implements a variety of models, including those described by Lee and Carter (1992), Brouhns, Denuit and Vermunt (2002), Cairns, Blake and Dowd (2006), Richards, Kirkby and Currie (2006), Richards and Currie (2009) and Currie (2010). Most of the basic model-fitting programs were written by Dr Iain Currie and then rewritten and extended by Stephen Richards to form a cohesive framework in a production environment. In addition to the basic models themselves, other features are implemented such as the over-dispersion parameter described by Djeundje and Currie (2009). As with the Longevitas system and mortalityrating.com, the user interface for the Projections Toolkit was written by Gavin Ritchie. As of February 2012 the Projections Toolkit was in use at a number of insurers, investment banks and consulting actuaries, mainly in the UK and the USA.

Chapter 20

Conclusions

Longevity risk may well be the defining actuarial challenge of the twenty-first century. More people are reaching retirement age and they are also living for longer. There is also a push from state provision of retirement income to private provision. The assessment of mortality differentials for underwriting will continue to develop as insurers seek new risk factors to sharpen their pricing and avoid anti-selection. This takes on extra urgency as EU law forbids insurers from using one of the simplest and most reliable factors — gender — from December 2012. Regulators will continue to demand the use of stochastic projections models in setting reserves as a means of acknowledging inherent uncertainty over the future path of mortality rates.

Longevity risk has a number of separate elements of interest to actuaries and their clients, including model risk, basis risk, parameter risk, idiosyncratic risk and concentration risk. Each of the six papers in the accompanying these takes a different facet of longevity risk and advances the modelling of it for actuarial work. The author has built upon existing theory and techniques, both from the actuarial sphere and from without. These techniques have been extended and applied to problems of liability management. In turn, other researchers have independently confirmed these results and built on them. In most cases the theory developed in the published papers has been implemented in software which is used by other actuaries in practical business situations.

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Appendix 1

Richards, S. J., Kirkby, J. G. and Currie, I. D. (2006) *The importance of year of birth in two-dimensional mortality data*, British Actuarial Journal, **12(I)**, 5–61.

Appendix 2

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Appendix 3

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Appendix 4

Richards, S. J. (2008) *Detecting year-of-birth mortality patterns with limited data*, Journal of the Royal Statistical Society, Series A, **171(1)**, 279–298.

Unlike the other five papers, the version contained here is not a facsimile of the original due to copyright restrictions.

Appendix 5

Richards, S. J. and Currie, I. D. (2009) *Longevity risk and annuity pricing with the Lee-Carter model*, British Actuarial Journal, **15(II)**, No. 65, 317–365.

Appendix 6

Richards, S. J. (2010) *A handbook of parametric survival models for actuarial use*, Scandinavian Actuarial Journal, DOI:10.1080/03461238.2010.506688.